The Classification of homotopy classes of bounded curvature paths: Towards a metric knot theory

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Figure: A river meander

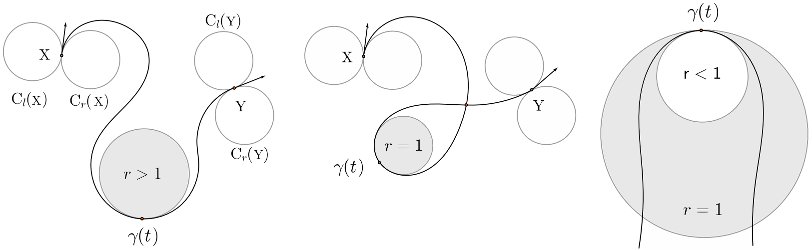
Figure: A windy road in the Andes

Definition of Bounded Curvature Path

Given $(x, X), (y, Y) \in UT\mathbb{M}$. An arc-length parametrised curve $\gamma : [0, s] \to \mathbb{M}$ connecting these points is a bounded curvature path if:

- γ starts at x ends at y with **fixed** tangent vectors X and Y respectively
- γ is C^1 and piecewise C^2
- $||\gamma''(t)|| \leq \kappa$, for all $t \in [0, s]$ when defined, $\kappa > 0$ a constant

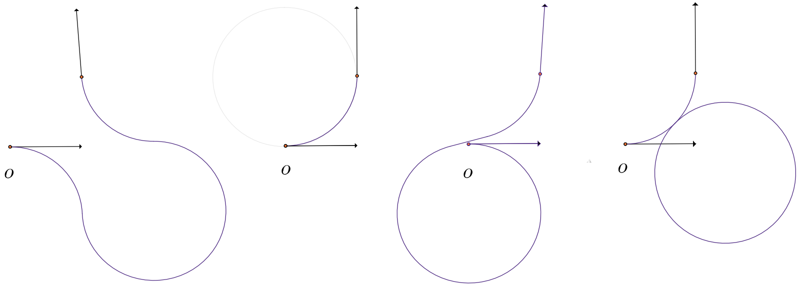
Examples



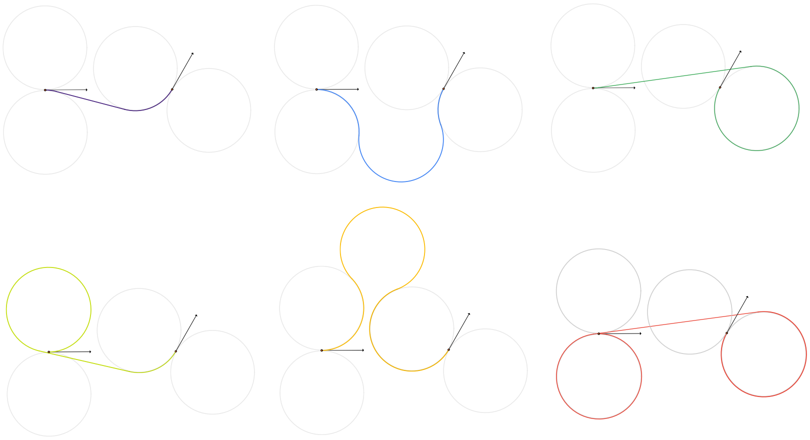
Lester E. Dubins, On plane curves with curvature Pacific J. Math. 11 (1961), no. 2, 471–481

"Here we only begin the exploration, raise some questions that we hope will prove stimulating, and invite others to discover the proofs of the definite theorems, proofs that have eluded us"

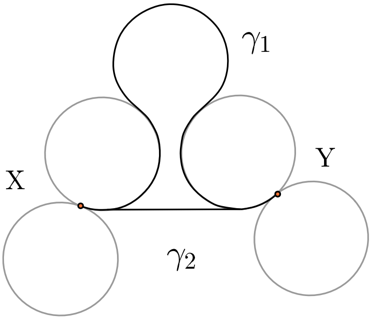
Length discontinuities

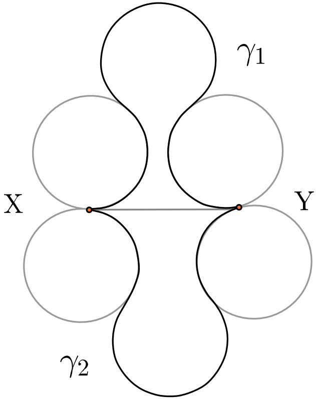


Existence of many local maxima



An interesting example





The space of bounded curvature paths

Given $\mathrm{X},\mathrm{Y} \in UT\mathbb{M}$, and a maximum curvature $\kappa > 0$.

The space of bounded curvature paths defined in $\mathbb{M}$ satisfying $\mathrm{X},\mathrm{Y} \in UT\mathbb{M}$ is denoted by $\Gamma(\mathrm{X},\mathrm{Y})$.

Definition

Given $\gamma, \eta \in \Gamma(\mathrm{X},\mathrm{Y})$. A bounded curvature homotopy between $\gamma : [0, s_0] \to \mathbb{M}$ and $\eta : [0, s_1] \to \mathbb{M}$ corresponds to a continuous one-parameter family of paths $\mathcal{H}_t : [0, 1] \to \Gamma(\mathrm{X},\mathrm{Y})$ such that:

- $\mathcal{H}_t(0) = \gamma(t)$ for $t \in [0, s_0]$ and $\mathcal{H}_t(1) = \eta(t)$ for $t \in [0, s_1]$.
- $\mathcal{H}_t(p) : [0, s_p] \to \mathbb{M}$ for $t \in [0, s_p]$ is an element of $\Gamma(\mathrm{X},\mathrm{Y})$ for all $p \in [0, 1]$.

Questions

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- What are the connected components in $\Gamma(\mathrm{X},\mathrm{Y})$?

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- What can we say about $\Gamma(\text{X},\text{Y})$ in punctured surfaces?
- What if the initial and final vectors are allowed to vary?
- What about $\Gamma(\text{X},\text{Y})$ for $\mathbb{M} = \mathbb{R}^3$?

Part I: Minimal length elements in $\Gamma(\mathrm{X},\mathrm{Y})$

A fragmentation of a bounded curvature path $\gamma : I \to \mathbb{M}$ corresponds to a finite sequence $0 = t_0 < t_1 \ldots < t_m = s$ of elements in I such that,

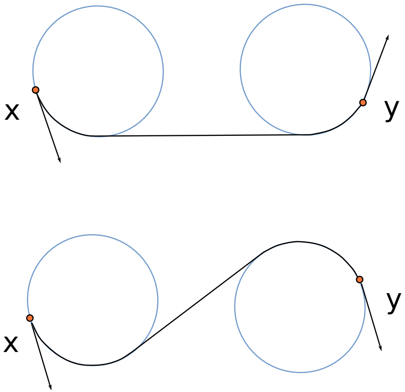
$$\mathcal{L}(\gamma, t_{i-1}, t_i) < r$$

with,

$$\sum_{i=1}^{m} \mathcal{L}(\gamma, t_{i-1}, t_i) = s$$

We denote by a fragment, the restriction of γ to the interval determined by two consecutive elements in the fragmentation.

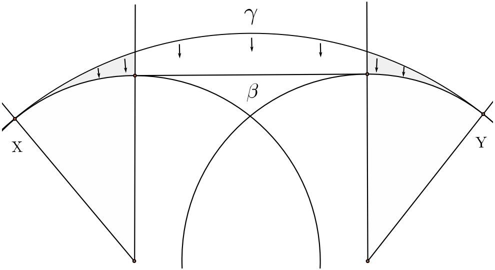
CSC paths



Observe that an arc of a circle can be left L or right R oriented.

Theorem

A fragment is bounded-homotopic to a CSC path.



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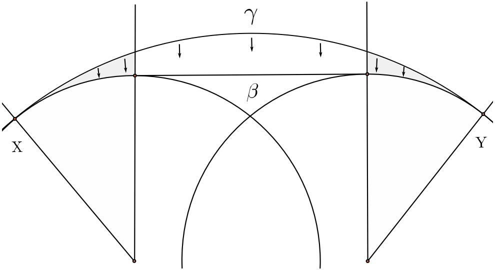
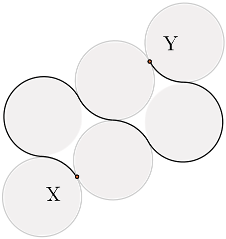


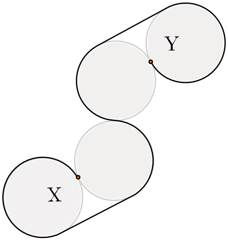
- The CSC path β is called replacement path.
- The length of β is at most the length of the fragment.

Complexity

A *cs* path is a concatenation of a finite number of line segments, or arcs of radius r circles.

The *complexity* of a cs path is number of line segments and circular arcs.



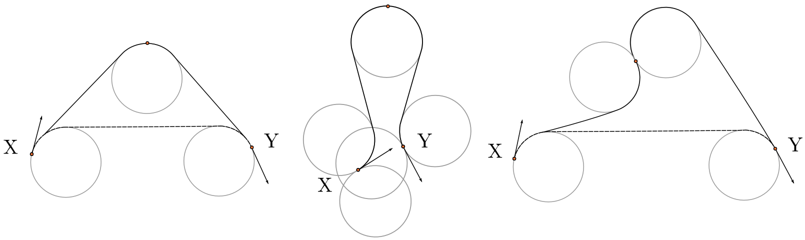


Theorem

Every bounded curvature path in $\Gamma(\mathrm{X},\mathrm{Y})$ can be altered to *cs* form (normalization), so that the path length does not increase.

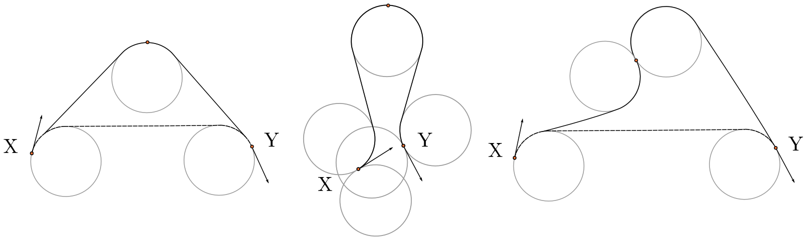
Proposition

Generic components are not paths of minimal length.



Theorem (global reduction)

A *cs* path with a generic component is bounded-homotopic to a *cs* path with less complexity without increasing its length.



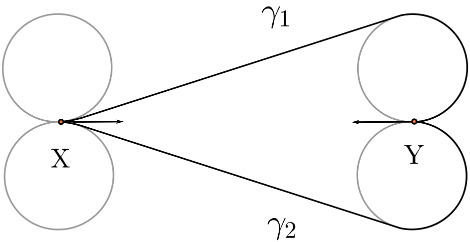
Theorem (Dubins)

Choose $\mathrm{X},\mathrm{Y} \in UT\mathbb{R}^2$ and a maximum curvature $\kappa > 0$. The minimal length bounded curvature path in $\Gamma(\mathrm{X},\mathrm{Y})$ is either a:

- CCC path having its middle component of length greater than πr or a
- CSC path where some of the circular arcs or line segments can have zero length

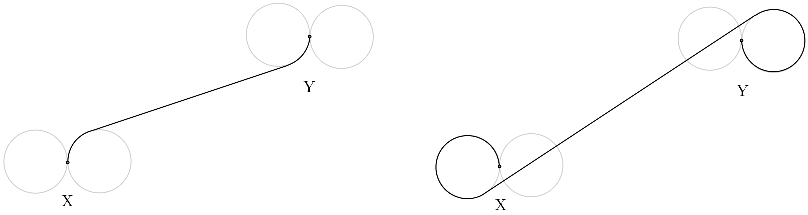
But want to find the minimal length elements in homotopy classes.

Are γ_1 and γ_2 in the same connected component?



Operations on *cs* paths: A RSL into a LSR

Are these two paths in the same homotopy class?



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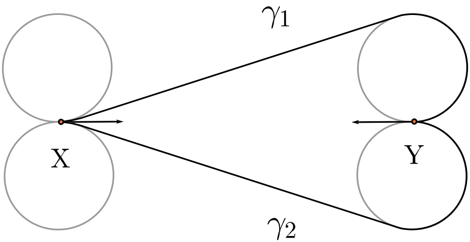
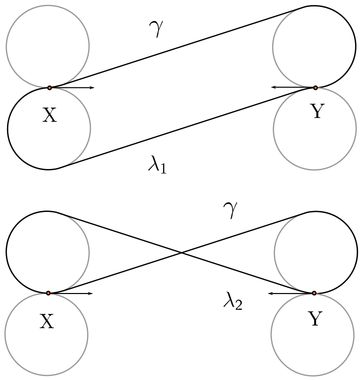


We want to make these paths closed paths.



Once we choose a closure path we stick with it!

Definition

Given $\mathrm{X},\mathrm{Y} \in UT\mathbb{M}$ together with a prescribed closure path λ.

$$\Gamma(n) = \{\gamma \in \Gamma(\mathrm{X},\mathrm{Y}) \mid \ T_\lambda(\gamma) = n, \ \ n \in \mathbb{Z}\}$$

Theorem: Minimal length elements in homotopy classes

Given X,Y $\in UT\mathbb{M}$ and $n \in \mathbb{Z}$. Then the minimal length bounded curvature path in $\Gamma(n)$ for $n \in \mathbb{Z}$ must be of the form:

- CSC or CCC
- $C^{\chi}SC$ or CSC^{χ} or $C^{\chi}CSC$
- $C^{\chi}CC$ or $CC^{\chi}C$

Here χ is the minimal number of crossings for paths in $\Gamma(n)$. In addition, some of the circular arcs or line segments may have zero length.

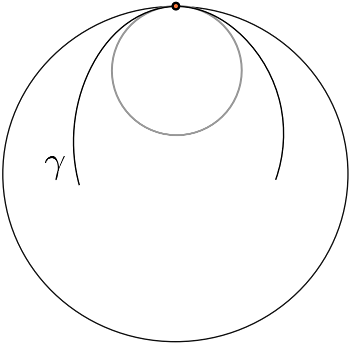
Dubins Explorer

Part II: Isotopy classes of bounded curvature paths

For certain $\mathrm{X},\mathrm{Y} \in UT\mathbb{M}$ a family of embedded bounded curvature paths get encapsulated in some regions in 2-space.

Curvature comparison lemma in 2-space

If a C^2 arc-length paramametrized curve $\gamma : [0, s] \to \mathbb{R}^2$ with $||\gamma''(t)|| \leq \kappa$ lies in a radius r disk D. Then either γ is entirely in $\partial(D)$, or the interior of γ is disjoint from $\partial(D)$.



Diameter lemma in 2-space

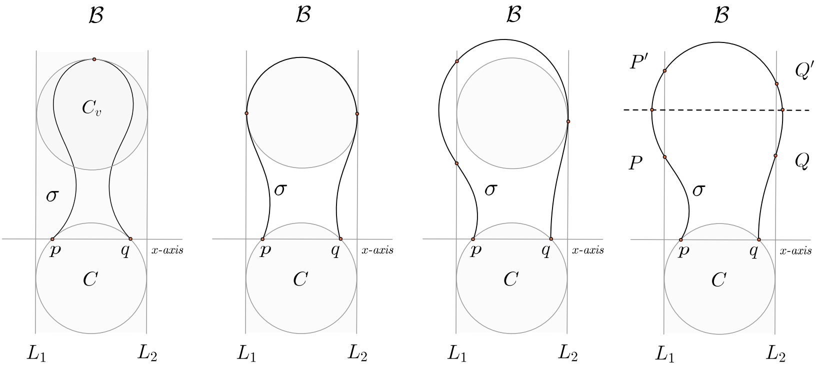
A bounded curvature path $\sigma : I \to \mathcal{B}$ where,

$$\mathcal{B} = \{(x, y) \in \mathbb{M} \,|\, -r < x < r \ , \ y \geq 0\}$$

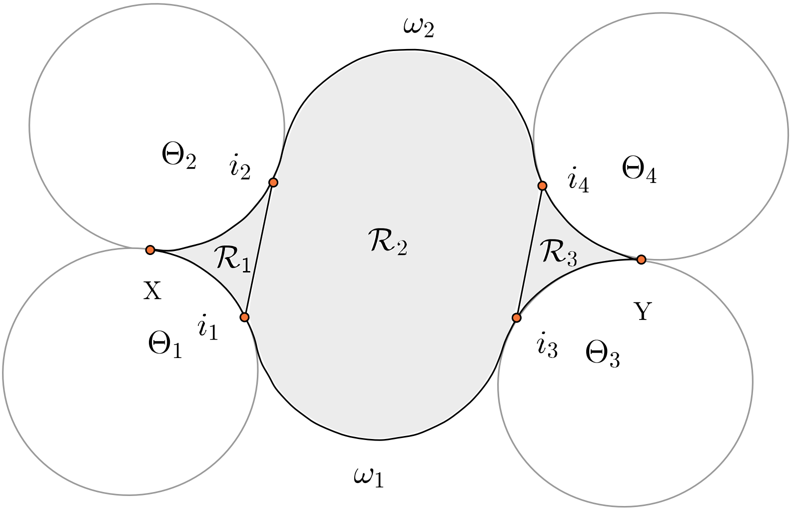
cannot satisfy both:

- $\sigma(0), \sigma(s)$ are points on the x-axis;
- If C is a radius r circle with centre on the negative y-axis and $\sigma(0), \sigma(s) \in C$, then some point in $Im(\sigma)$ lies above C.

Diameter lemma in 2-space

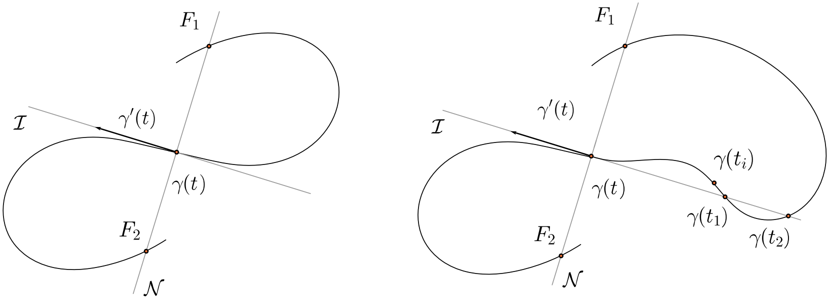


Theorem: $diam(\Omega) < 4r$

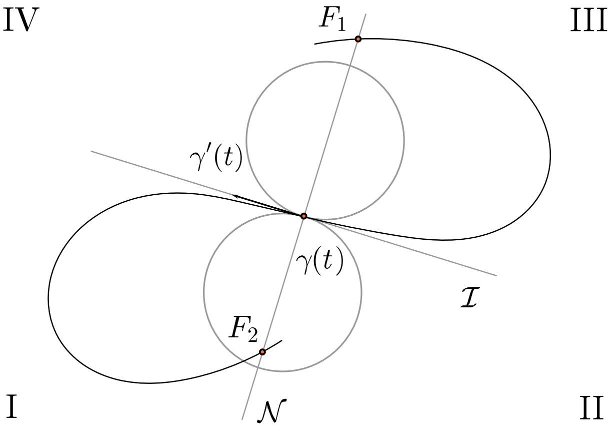


Definition

A maximal inflection with respect to $\mathrm{x} \in T\mathbb{M}$ is a minimum value of the turning map $\tau : I \to \mathbb{R}$



S-lemma



Theorem

Embedded bounded curvature paths in Ω cannot be made bounded-homotopic to paths with self intersections.

Embedded bounded curvature paths in Ω get trapped in Ω.

Classification of homotopy classes of bounded curvature paths

Given $\mathrm{X}, \mathrm{Y} \in UT\mathbb{M}$ where $\mathbb{M} = \mathbb{H}$ or $\mathbb{R}^2$ we have that:

$$\Gamma(\mathrm{X}, \mathrm{Y}) = \bigcup_{n \in \mathbb{Z}} \Gamma(n) \tag{1}$$

If $\mathrm{X}, \mathrm{Y} \in UT\mathbb{M}$ carries a region Ω, then $\Gamma(k)$ consist of two homotopy classes:

- one of embedded paths (isotopy class);
- the other consists of paths that wander over the plane

κ-constrained curves

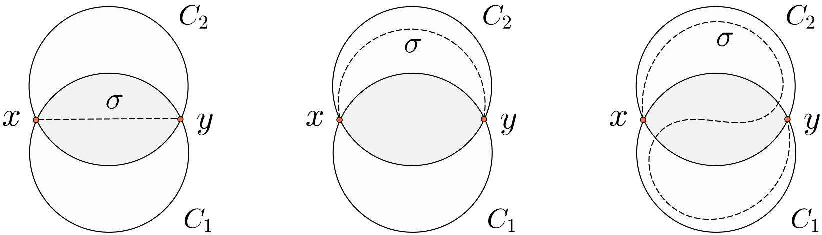
An arc-length parameterised plane curve $\sigma : [0, s] \to \mathbb{R}^2$ is called a κ-constrained curves if:

- σ is C^1 and piecewise C^2;
- $||\sigma''(t)|| \leq \kappa$, for all $t \in [0, s]$ when defined, $\kappa > 0$.

The space of κ-constrained curves connecting x to y is denoted by $\Sigma(x, y)$.

Example and non examples

Here $d(x,y) < 2r$



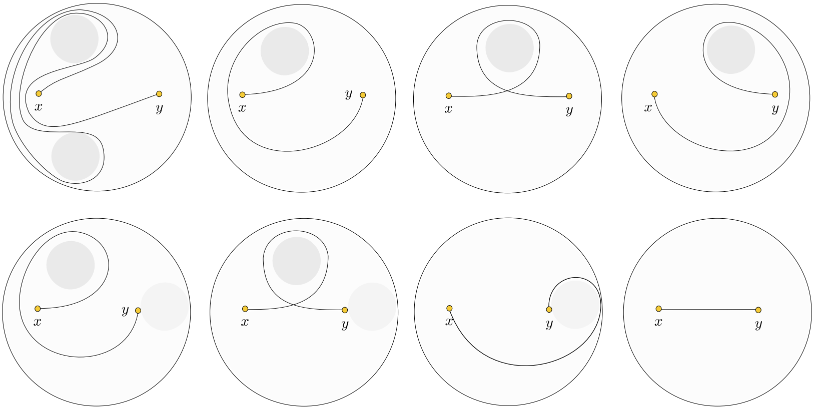
Classification of homotopy classes of κ-constrained curves

Choose $x, y \in \mathbb{M}$. Then:

$$|\Sigma(x,y)| = \begin{cases} 1 & d(x,y) = 0 \\ 2 & 0 < d(x,y) < 2r \\ 1 & d(x,y) \geq 2r \end{cases}$$

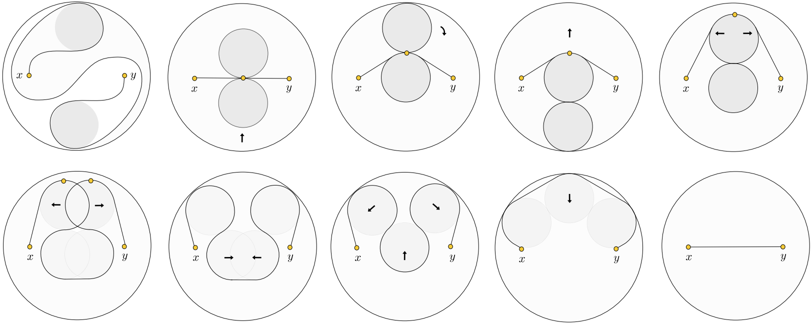
Work in progress: κ-constrained curves in a disk

$d(x, \partial D) < 2r$ and $d(y, \partial D) < 2r$



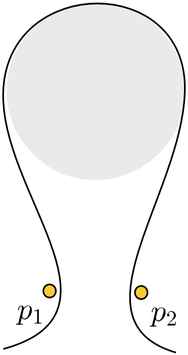
$d(x, \partial D) < 2r$ and $d(y, \partial D) \geq 2r$

Deformations of κ-constrained curves in a disk



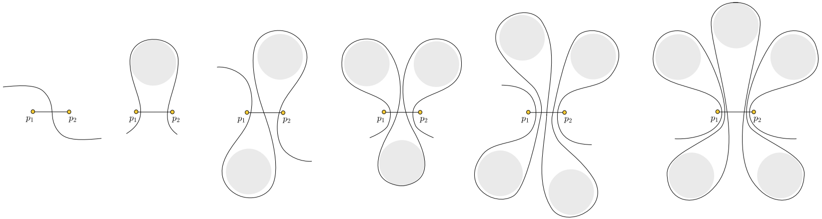
True for sufficiently large disk.
The radius of D is an important parameter.

A curve in between punctures

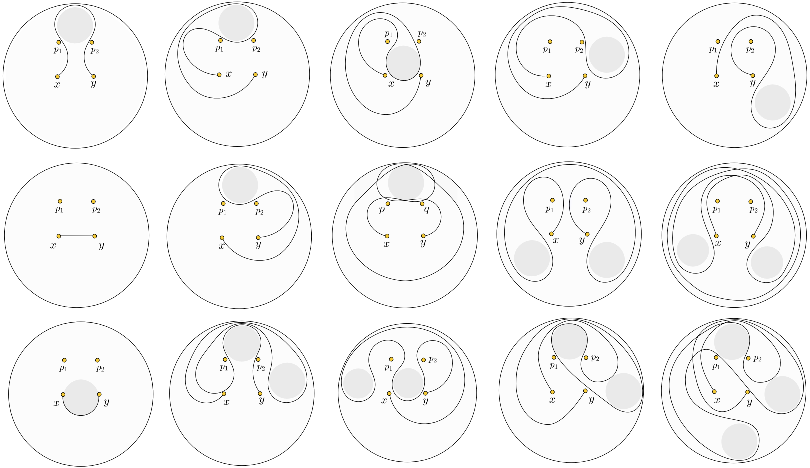


Here $d(p_1, p_2) < 2r$ with curvature bound $\kappa = 1/r$

κ-constrained curves in between punctures



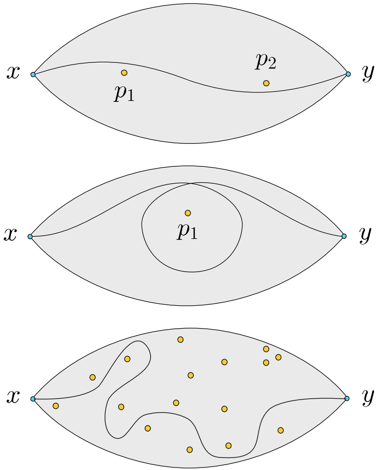
κ-constrained curves in a punctured disk



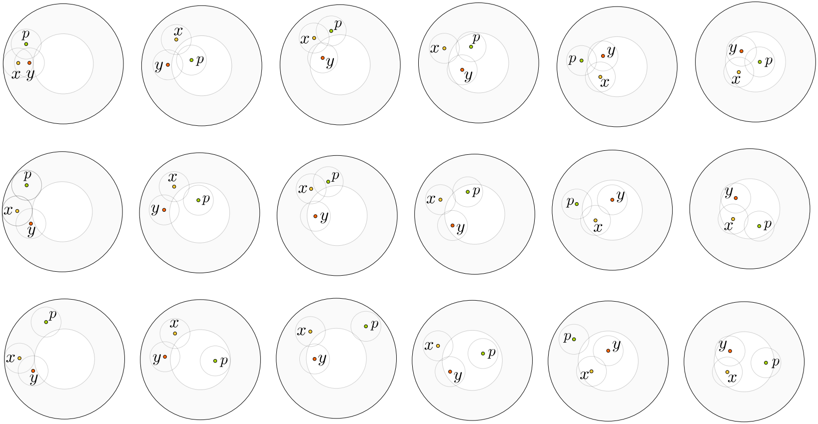
$d(x, y) < 2r$; $d(p_1, p_2) < 2r$; $d(p_1, \partial D) < 2r$; $d(p_2, \partial D) < 2r$

κ-constrained curves in between punctures

Here $d(x, y) < 2r$



Configuration of punctures p_i, x and y in D

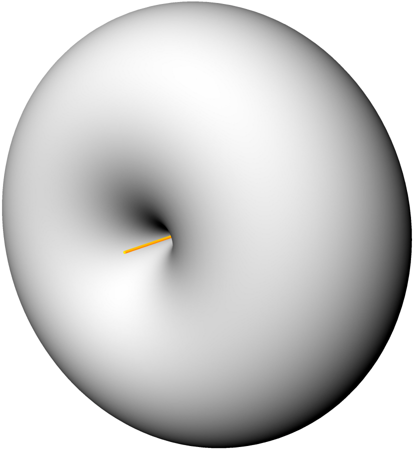


What about bc paths in dimension 3?

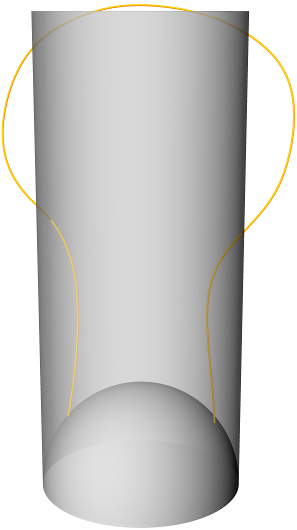
The only result known is due to H. Sussmann in 1995.

He characterised the length minimisers bounded curvature paths in $\mathbb{R}^3$.

A pinched torus is a local barrier for deformations

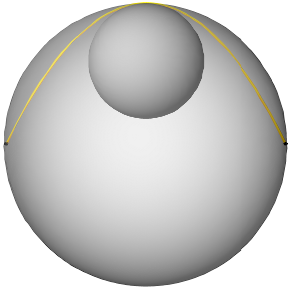


Lemma tube (analogous to Lemma band)



Comparison lemma (analogous to the 2-dimensional case)

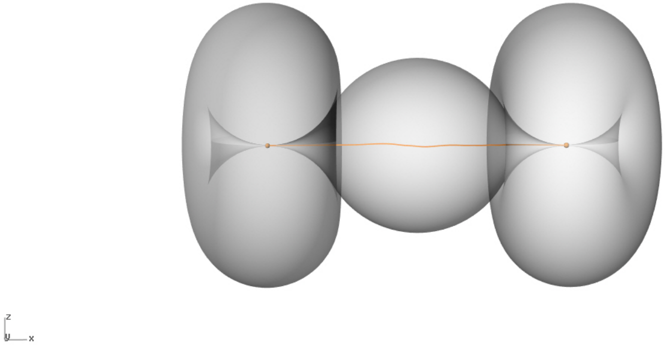
If a C^2 arc-length paramametrized curve $\gamma : [0, s] \to \mathbb{R}^3$ with $||\gamma''(t)|| \leq \kappa$ lies in a radius r ball B. Then either γ is entirely in $\partial(B)$, or the interior of γ is disjoint from $\partial(B)$.



Theorem: Isotopy condition for bounded curvature paths

An embedded bounded curvature path in $\Omega \subset \mathbb{R}^3$:

- it cannot be deformed to a path outside of the region Ω;
- it cannot be locally deformed to a path with a self-intersection.



What about physical knots?

- There are many models and approaches to study physical knots.

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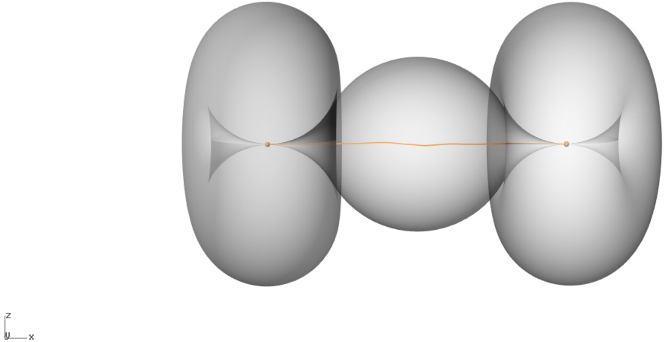
- A bounded curvature knot is a piecewise C^2 embedding of S^1 in $\mathbb{R}^3$ satisfying a prescribed bound on curvature.

Isotopy condition for bounded curvature knots

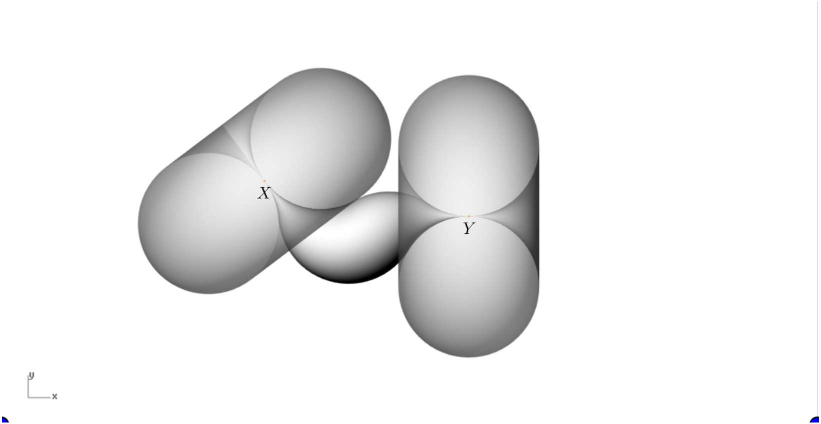
With the “isotopy condition for bc paths” we guarantee that knots satisfying a bound on curvature may be deformed without violating the curvature bound –while the knots remain in the same isotopy class.



Remarks

- Fragmentation process for bc knots
- Existence bc knots in each isotopy class
- Each isotopy class of bc knots may be characterized by the length of the length minimiser and also the number of pieces of type C or S (complexity of the knot).

Example of a 3D trapping region



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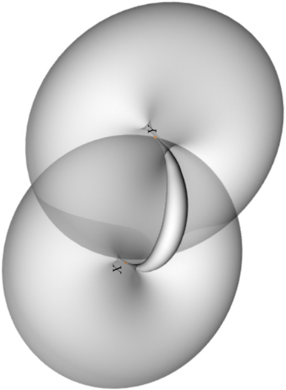
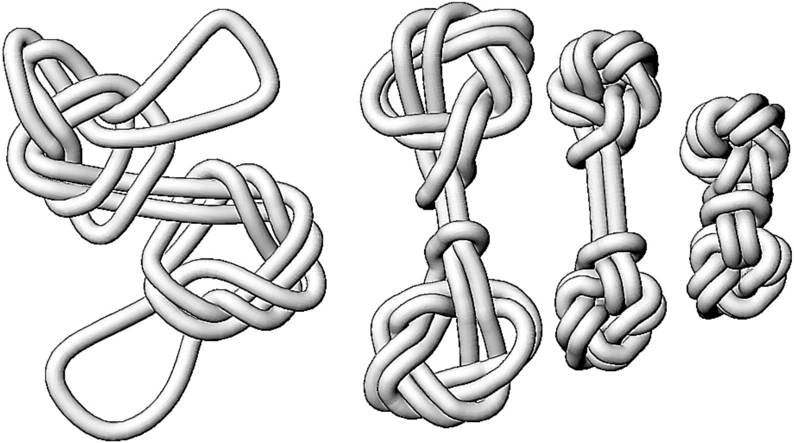


A conjectural Gordian unknot by Pieransky



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