Discrete geometry of surfaces

towards the filling area conjecture

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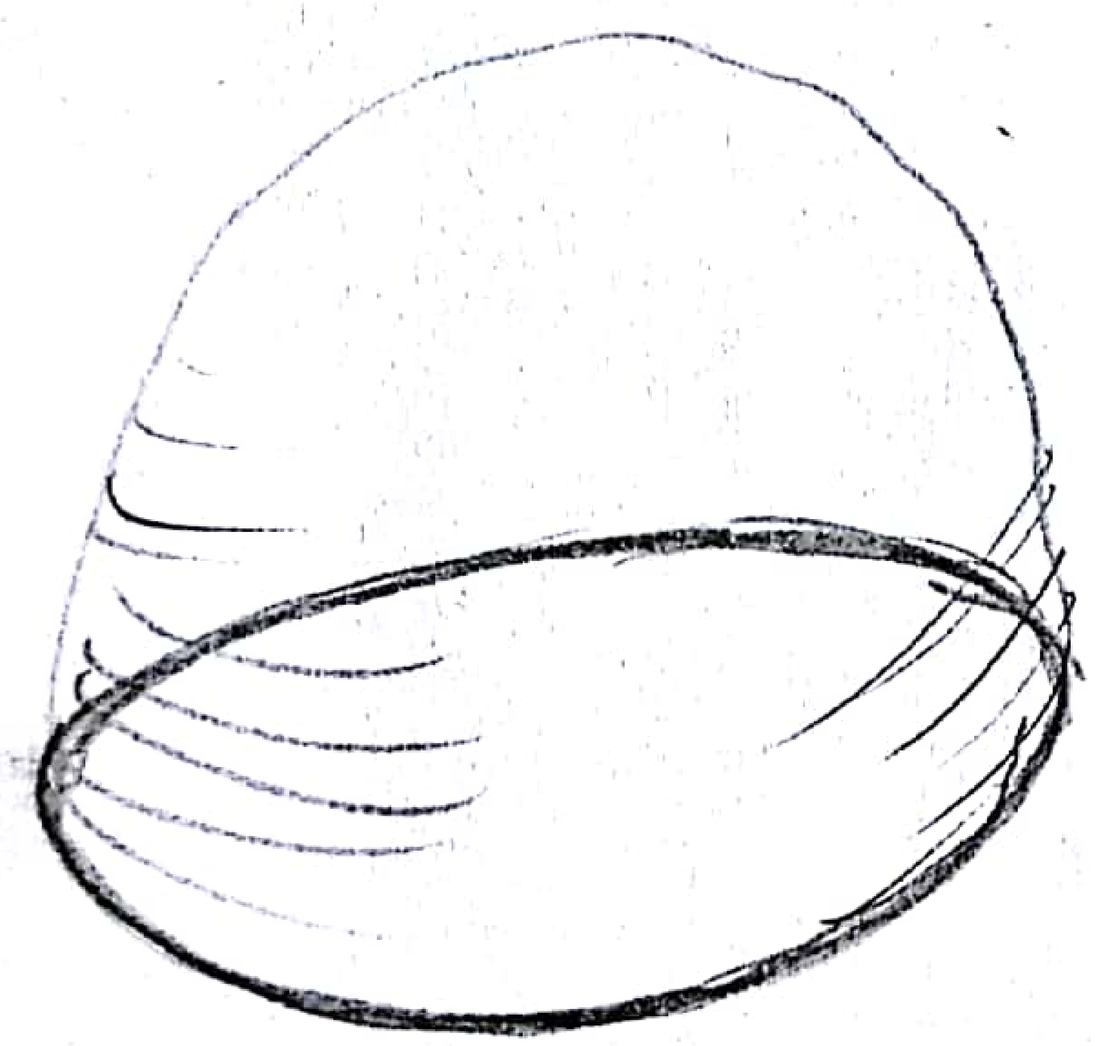
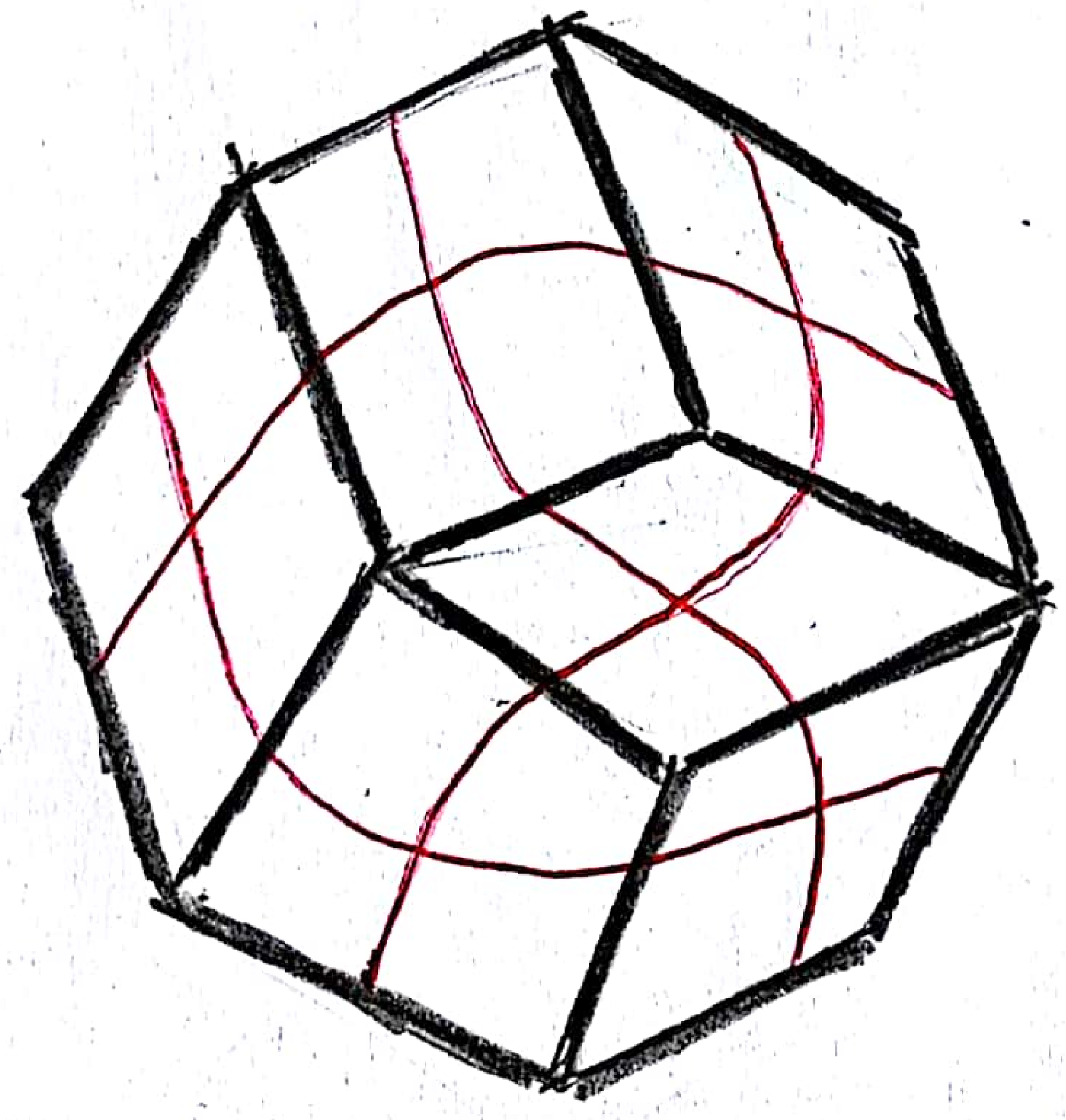
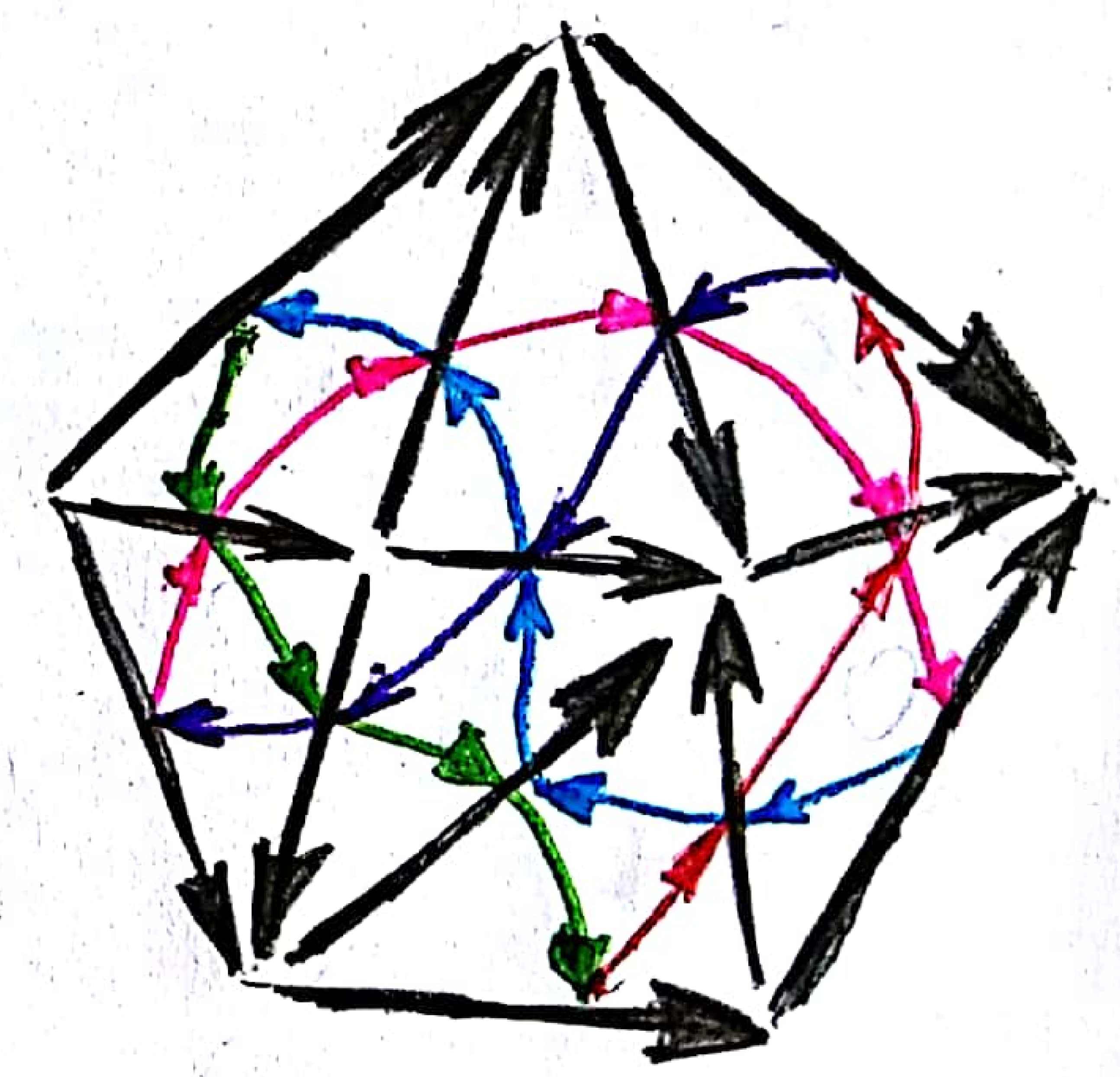






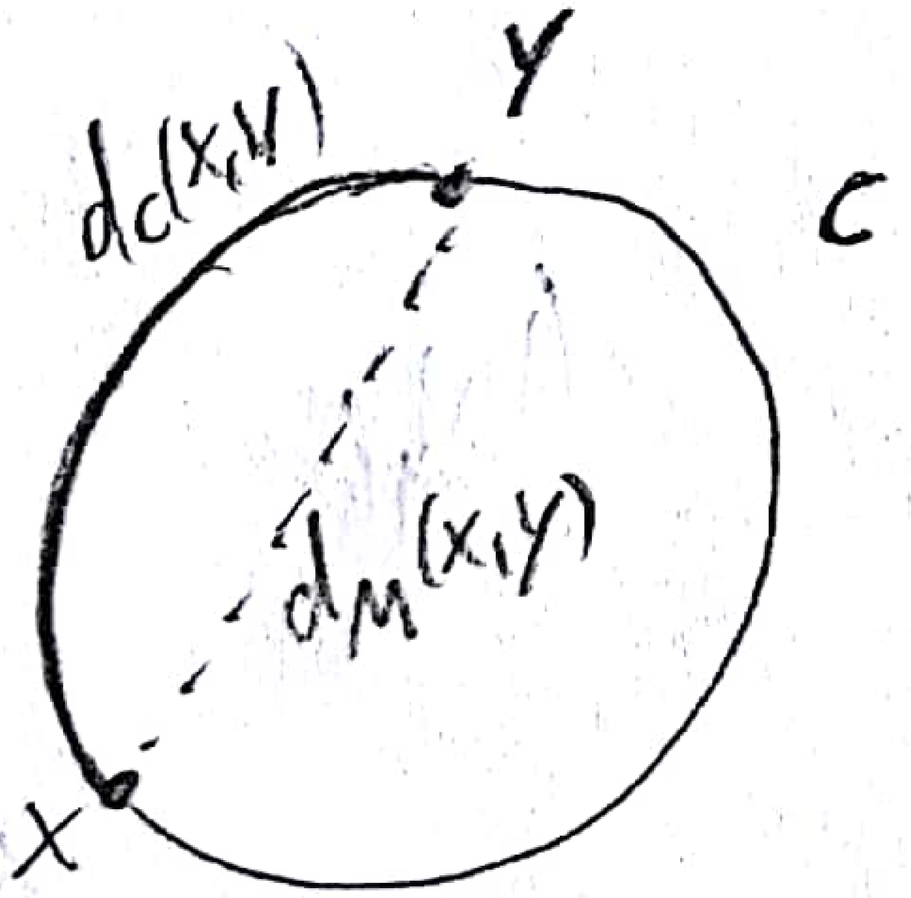
Problem: Given a closed curve C of certain length, find the minimum (or infimum) possible area of an isometric (b) filling (a) of C,

that is; a compact surface M such that

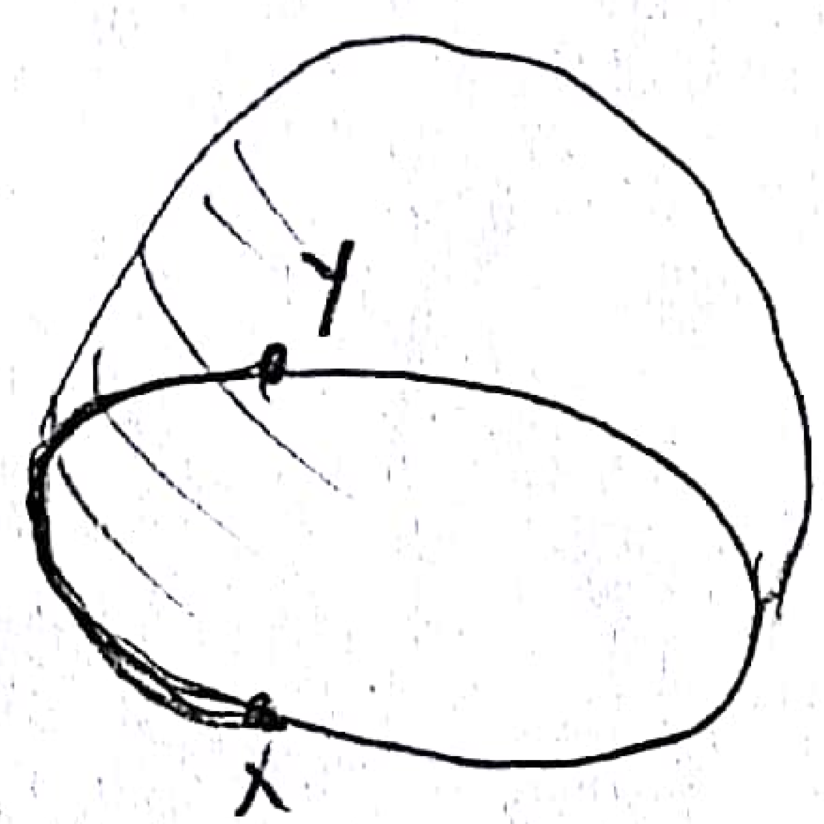
(a) $\partial M = C$

(b) $\underbrace{d_M(x,y)}_{:= \inf_{\substack{x \xrightarrow{\gamma} y \\ \text{curve along } M}} \mathrm{Len}(\gamma)} = d_C(x,y)$

Examples:



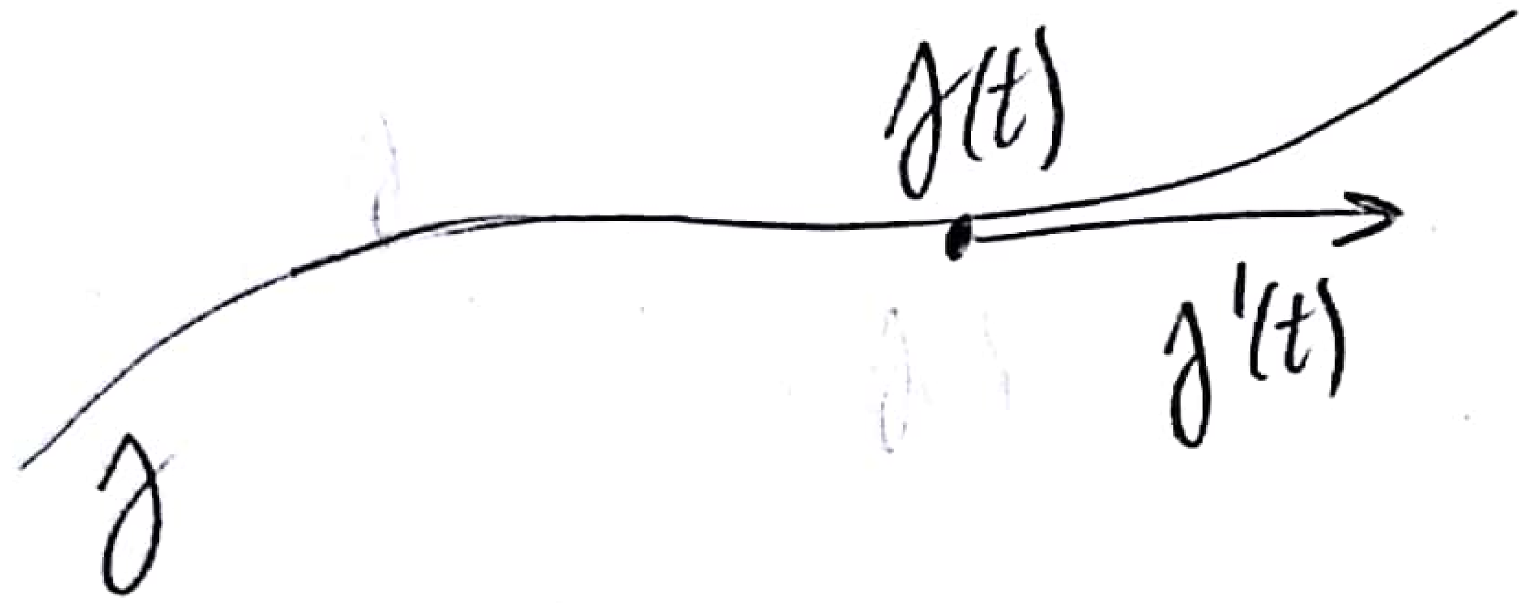
flat disk is a non-isometric filling



hemisphere is an isometric filling

Question: Is the hemisphere the minimum isometric filling?

M is a smooth surface that has a metric, to define length and area.



$$\mathrm{Len}(\gamma) := \int \|\gamma'(t)\|_{\gamma(t)}\, dt$$

If the metric is Riemannian, at each point $x \in M$ we have an Euclidean norm.

$$\|-\|_x : T_xM \longrightarrow [0,+\infty)$$

comes from an inner product g_x

$$\|v\|_x = \sqrt{g_x(v,v)}.$$

More generally, at each $x \in M$ we can have any directed norm.

$$\|-\|_x : T_xM \longrightarrow [0,+\infty).$$

$\|v+w\| \leq \|v\| + \|w\|$

$\|\lambda v\| = \lambda \|v\|$ if $\lambda \geq 0$

$\|v\| > 0$ if $v \neq 0$

$\cancel{\|-v\| = \|v\|}$

The function $F : TM \longrightarrow [0,+\infty)$

$$\underset{M}{(x}, \underset{T_xM}{v)} \longmapsto F_x(v) = \|v\|_x,$$

is required to be continuous and called a Finsler metric.

The norm can be given by its dual unit ball $B_x^* \subseteq (T_xM)^*$ compact, convex nghbd of origin

$$\|v\|_x = \max_{\varphi \in B_x^*} \varphi(v).$$

Area of a Finsler surface

For our purposes,

- a Riemannian surface can be approximated by a polyhedral surface made of Euclidean triangles
- a Finsler surface made of triangles cut from different normed planes.

How much area should each piece contribute?

Holmes–Thompson area

$$\mathrm{Area}_{UHT}(M,F) = \underbrace{|M|}_{\text{standard area in } \mathbb{R}^2} \cdot \underbrace{|B^*|}_{\text{standard area in } (\mathbb{R}^2)^* = \mathbb{R}^2} \cdot \underbrace{\cancel{\tfrac{1}{\pi}}}_{\text{normalization constant}}$$

M: piece of normed plane $(\mathbb{R}^2, \|\cdot\|)$

$\frac{1}{\pi}$: normalization constant to match standard definition on Euclidean plane (and Riemannian surfaces).

un-normalized

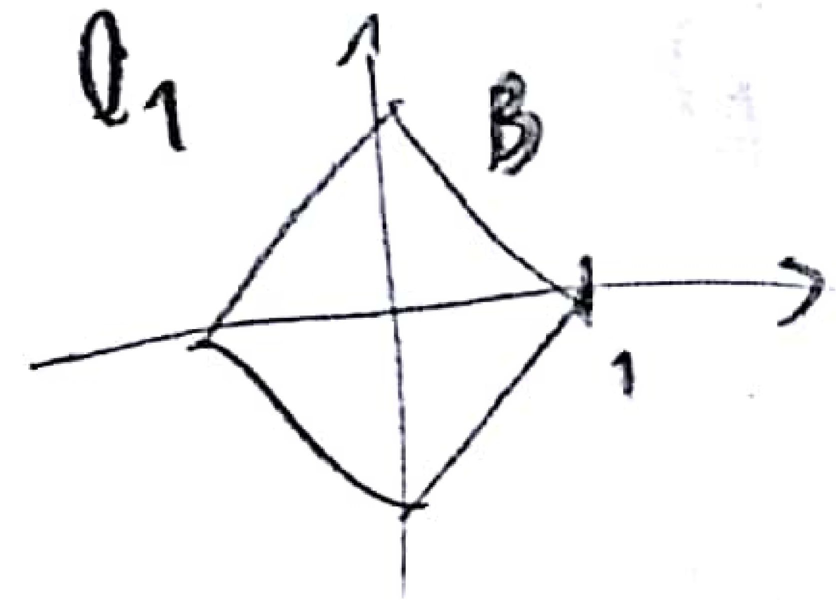
Examples:

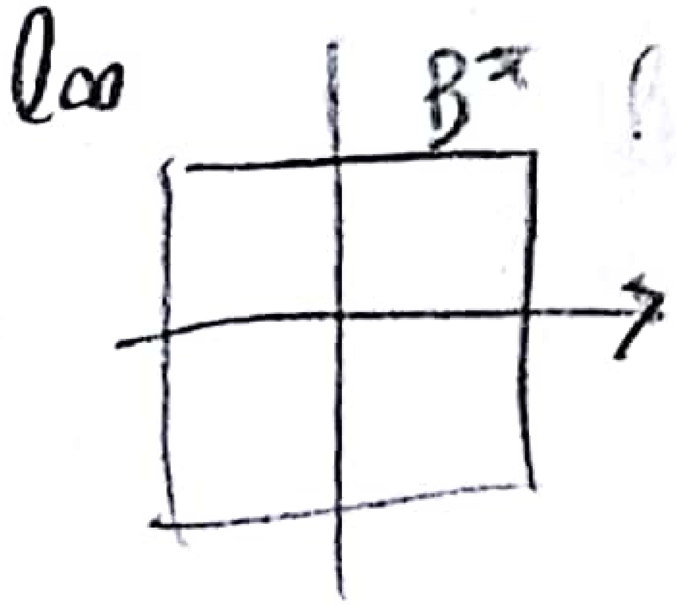
$$\mathrm{Area}_{UHT}\begin{pmatrix}\text{Euclidean} \\ \text{disk of} \\ \text{perimeter } 2L\end{pmatrix} = \pi r^2 \cdot \pi = L^2.$$

radius $r = \frac{L}{\pi}$.

$$\mathrm{Area}_{UHT}\begin{pmatrix}\text{Euclidean} \\ \text{hemisphere} \\ \text{of perim. } 2L\end{pmatrix} = 2L^2$$

$$\mathrm{Area}\begin{pmatrix}\text{square} \\ [0,L]\times[0,L] \\ \text{in } \ell_1 \text{ plane}\end{pmatrix} = L^2 \cdot |B^*| = 4L^2.$$





Back to the filling area problem ... what is known?

<u>Gromov</u> (1983), "Filling Riemannian manifolds".

– Posed question, allowing <u>Riemannian metrics</u> on M

– Conjectured minimality of hemisphere among orientable fillings.

(now called "filling area conjecture", or FAC)

– Proved strict minimality of hemisphere among Riemannian <u>disks</u>.

(Later, Bangert-Croke-Ivanov-Katz proved minimality among fillings of genus ≤ 1. )

<u>Ivanov</u> (2001)

– Extended question admitting <u>Finsler metrics</u> on M.

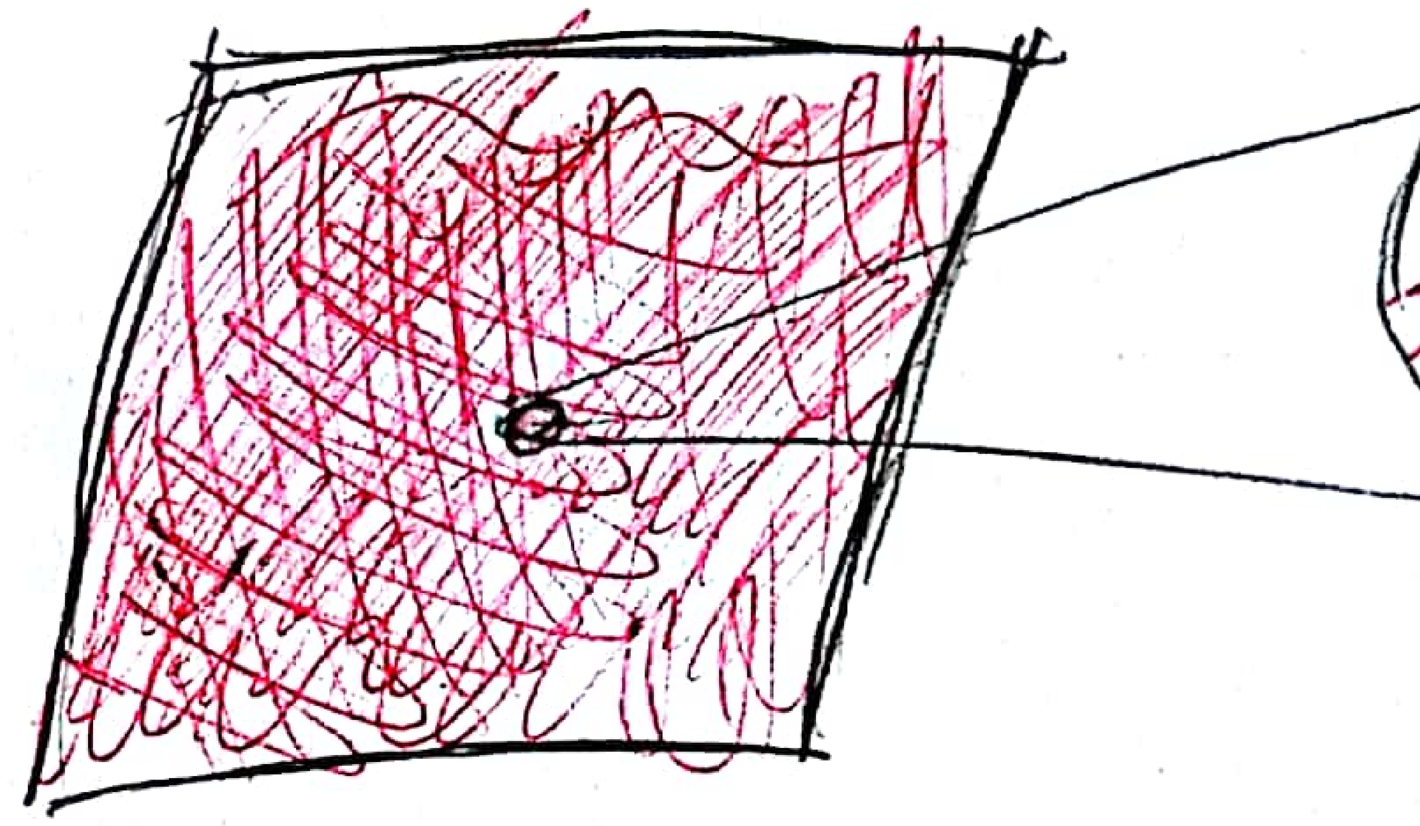
– Proved minimality of hemisphere still holds among <u>Finsler disks</u>.

– But there are <u>many</u> Finsler disks that attain <u>the same area</u> as the hemisphere.

Let's call them <u>Finsler hemispheres</u>. using Holmes-Thompson definition

<u>Burago - Ivanov</u> (2002) constructed "rational" Finsler counterexample M, with $\partial M = k \cdot C$, $\text{Area}(M) < k \cdot \text{Area}(\text{hemisph})$ $k=10$

Imagine that you zoom into the surface M and discover that the metric is actually a system W of curves, called walls.



$\mathrm{Len}(\gamma) =$ # of crossings of γ with W

$\mathrm{Area}(M) =$ # of self-crossings of W.

(if we use appropriate units ... More precisely, if we define $\mathrm{Area}_W(M) :=$ # of self-crossings, then $\mathrm{Area}_{HT} = 4 \cdot \mathrm{Area}_W$.)

(Always assume the curves $W, \gamma, \partial M$ are properly immersed and in general position.
- transverse intersections
- no triple crossings.)

<u>Problem:</u> If (M, W) fills isometrically its boundary (of length $2n$), how small can its area be?

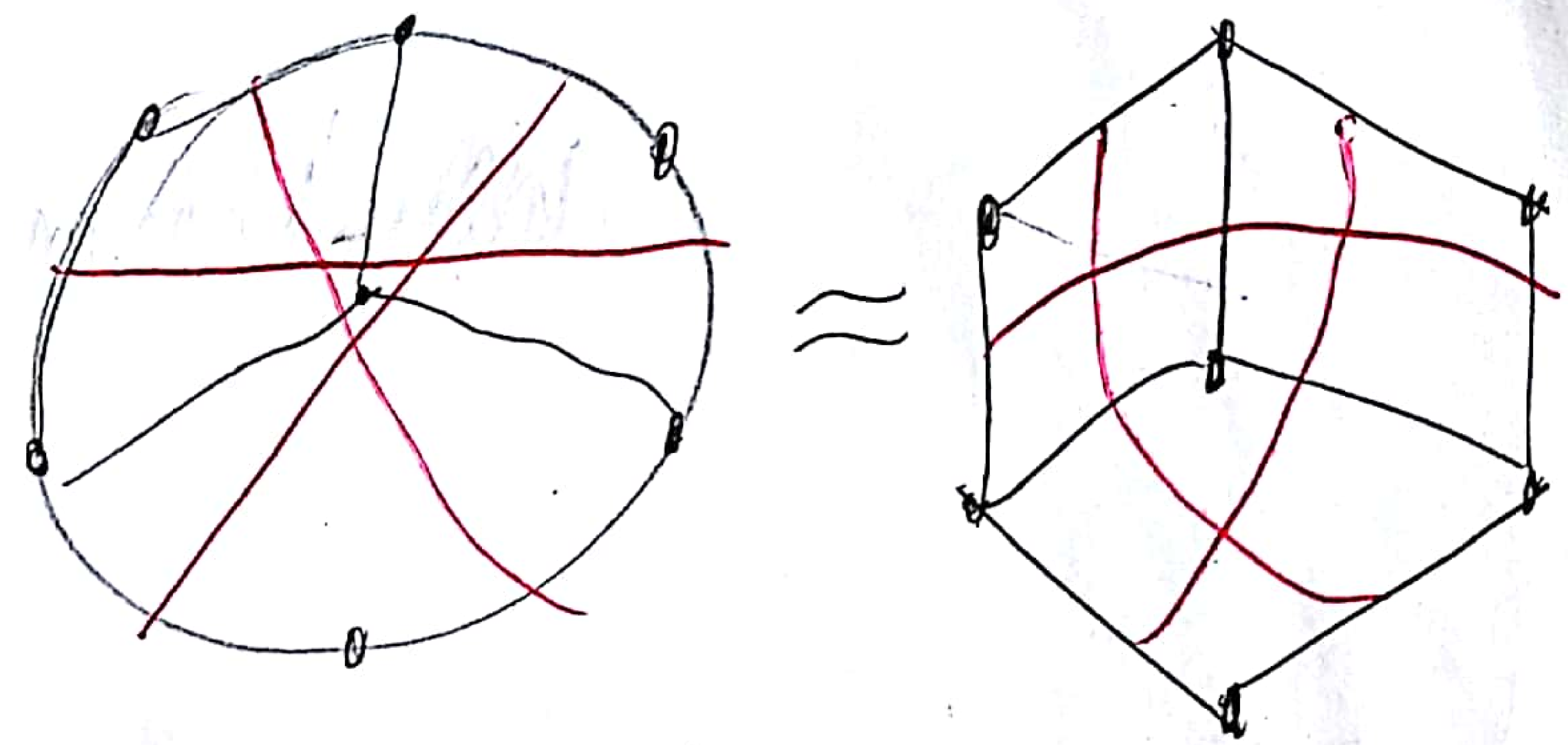
<u>Conjecture</u> (discrete FAC): Min area $= \dfrac{n(n-1)}{2}$.

- Can be proved when M is a <u>disk</u> or <u>Möbius band</u>.
- <u>Equivalent</u> to continuous conjecture (for self-reverse Finsler metrics), for each topology of M.

Purely combinatorial version :

The dual of a walled surface (M, W)

wallsystem, considered as embedded graph



is a square-celled combinatorial surface.

Discrete FAC Says :

To fill isometrically a cycle graph C_{2n} (of even length $2n$), we need $\geq \frac{n(n-1)}{2}$ cells.

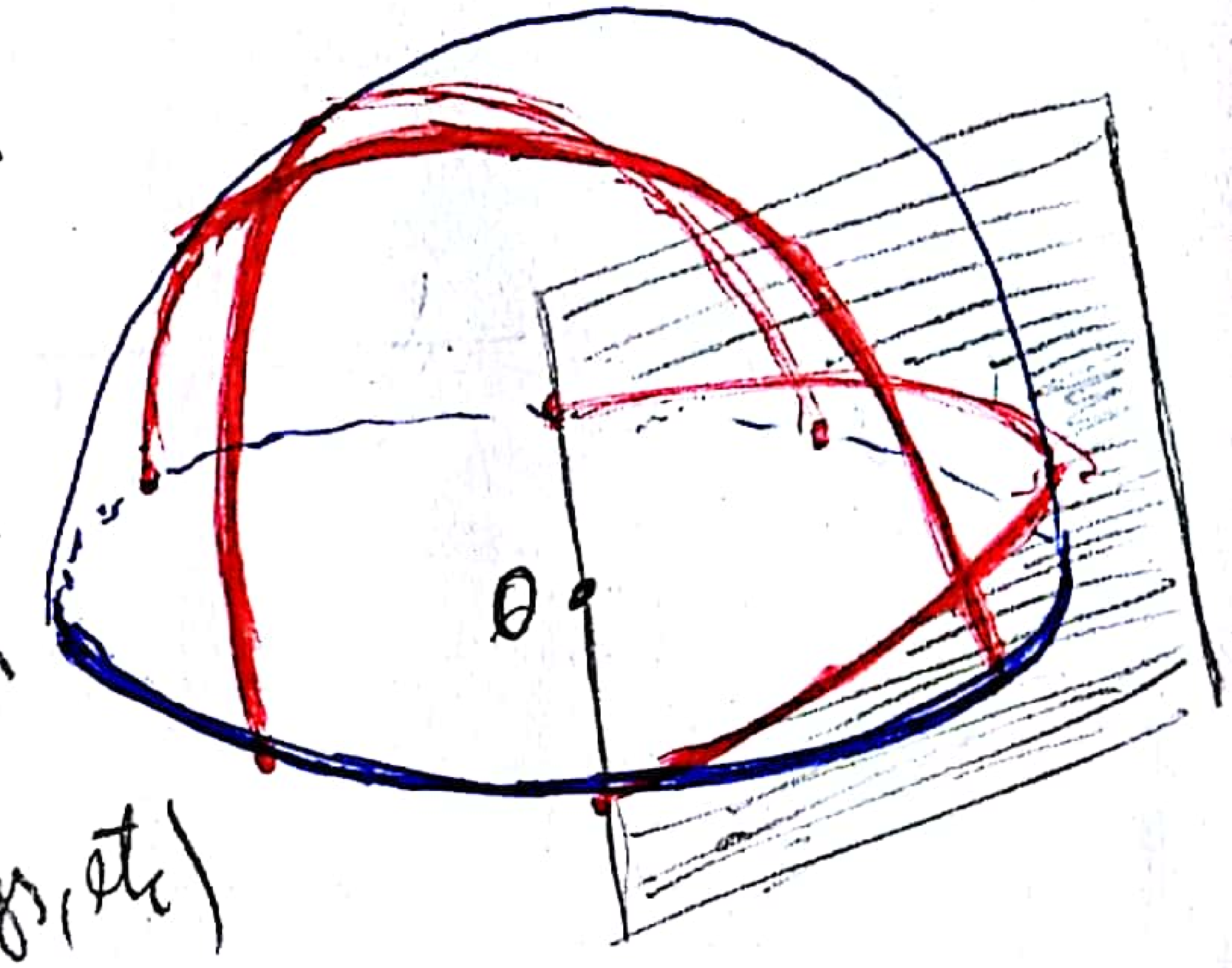
distance between boundary vertices is measured along the skeleton graph ; it should not be smaller than the distance along boundary cycle.

How to construct an isometric filling with Area $= \frac{n(n-1)}{2}$.

surface $M =$ Upper hemisphere (of standard sphere $S^2 \subseteq \mathbb{R}^3$ centered at O).

wallsystem W: obtained by intersecting with M n random planes through origin (independent, with uniform distribution)

almost surely W is a wallsystem (no triple crossings, etc)



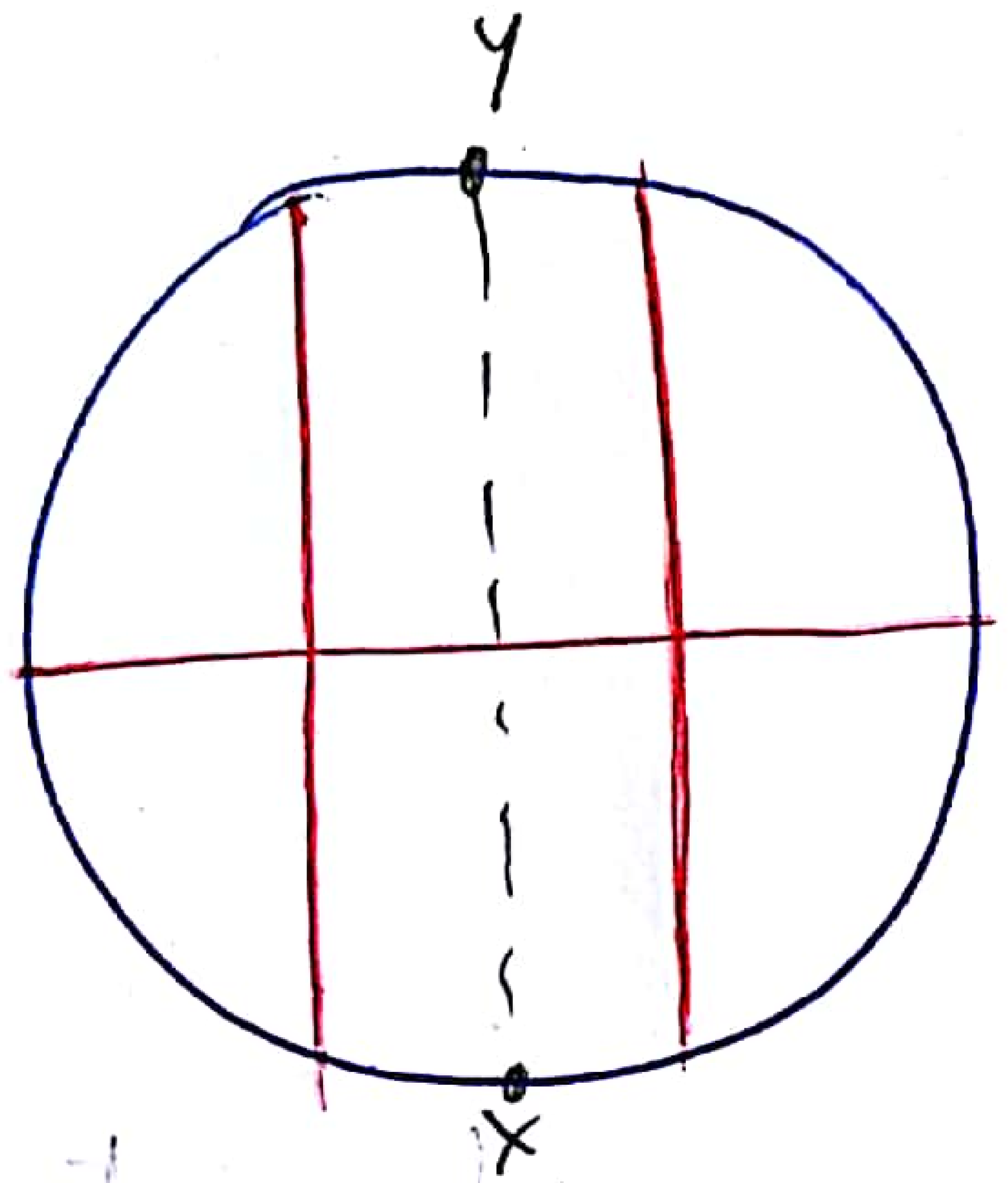
<u>Isometric filling</u> because

$$d_C(x,y) = d_M(x,y) = \#\text{ planes that separate } x \text{ from } y \qquad \forall x,y \in C = \partial M.$$

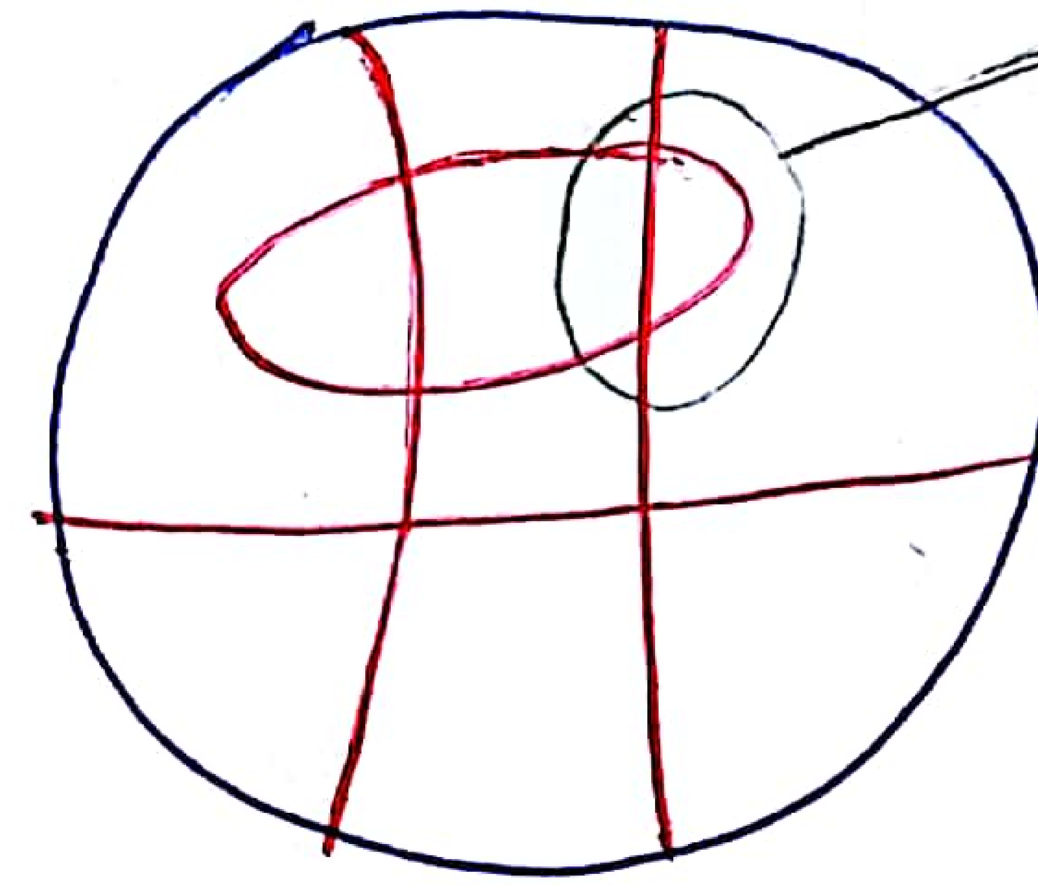
Has area $= \frac{n(n-1)}{2}$ because each pair of planes crosses once in M.

Examples:

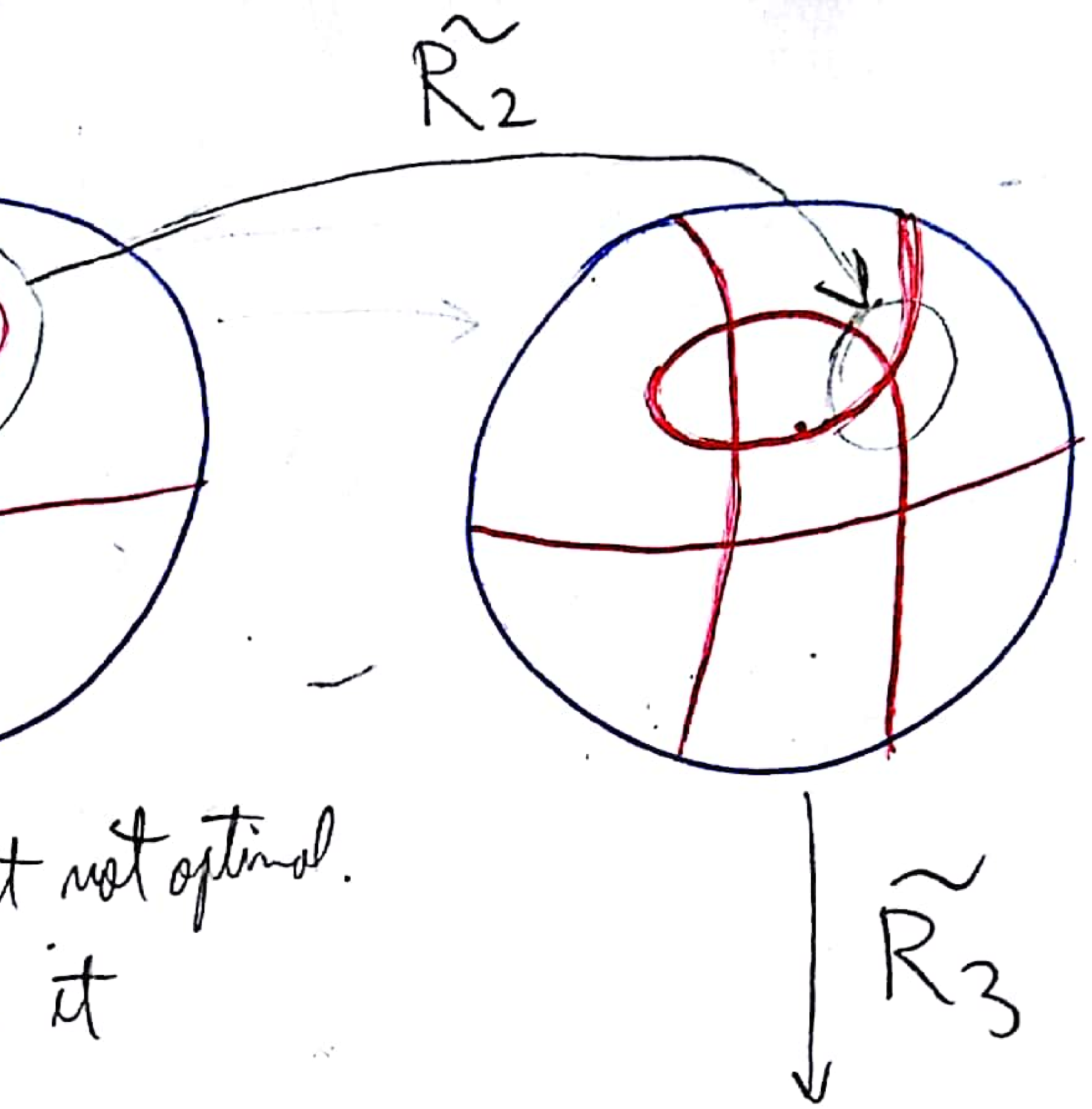
$n=3$

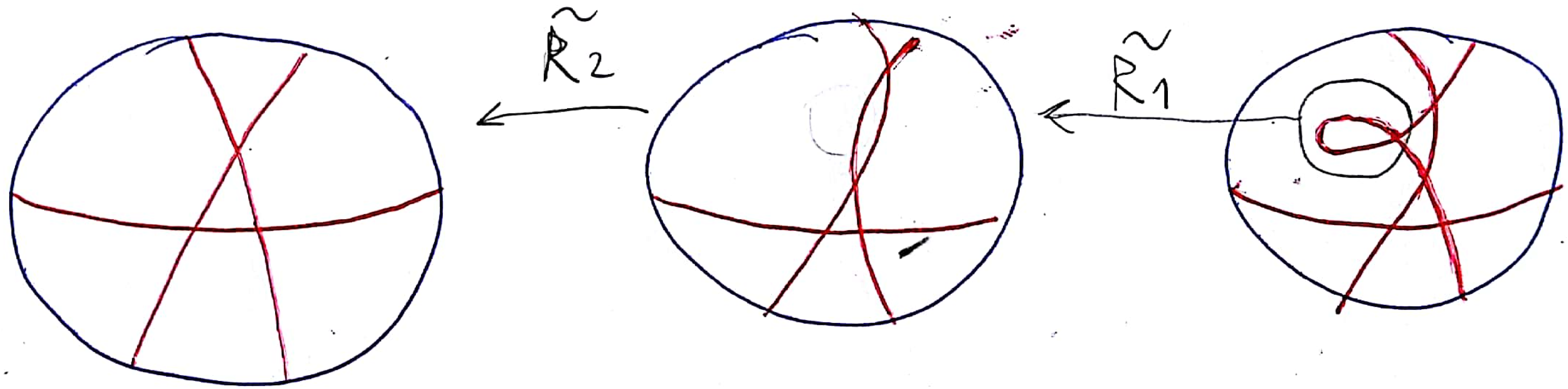


not isometric: $d_M(x,y)=1$
$d_C(x,y)=3$



isometric, but not optimal.
Let's improve it





Has $\frac{n(n-1)}{2} = 3$ crossings.

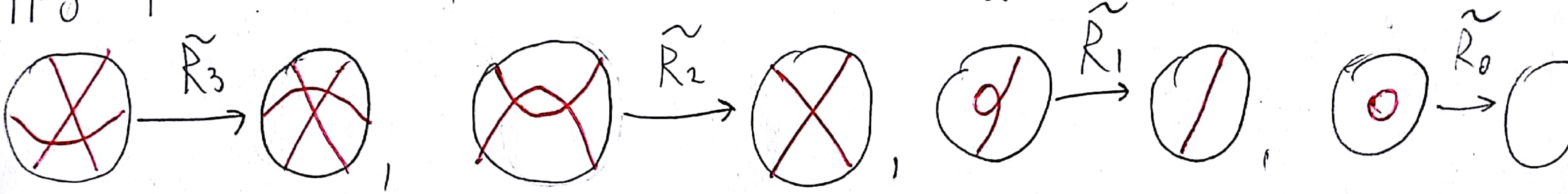
optimal?

Proof that area $\geq \frac{n(n-1)}{2}$ if M is a disk

- Let (M, W) be an isometric filling of its boundary, of length $2n$.

Schrijver 1991

- Apply operations $\tilde{R}_i$, that reduce area without affecting boundary distances.



- Eventually, the wallsystem W becomes a pseudoline arrangement W',

the walls are n simple arcs that cross each other at most once.

but it still fills isometrically.

- Show that on any pseudoline arrangement, $d(x,y)$ = # walls that separate x from y.

In particular, each pair of opposite points $x, y \in \partial M \setminus W'$ must be separated by all n walls.

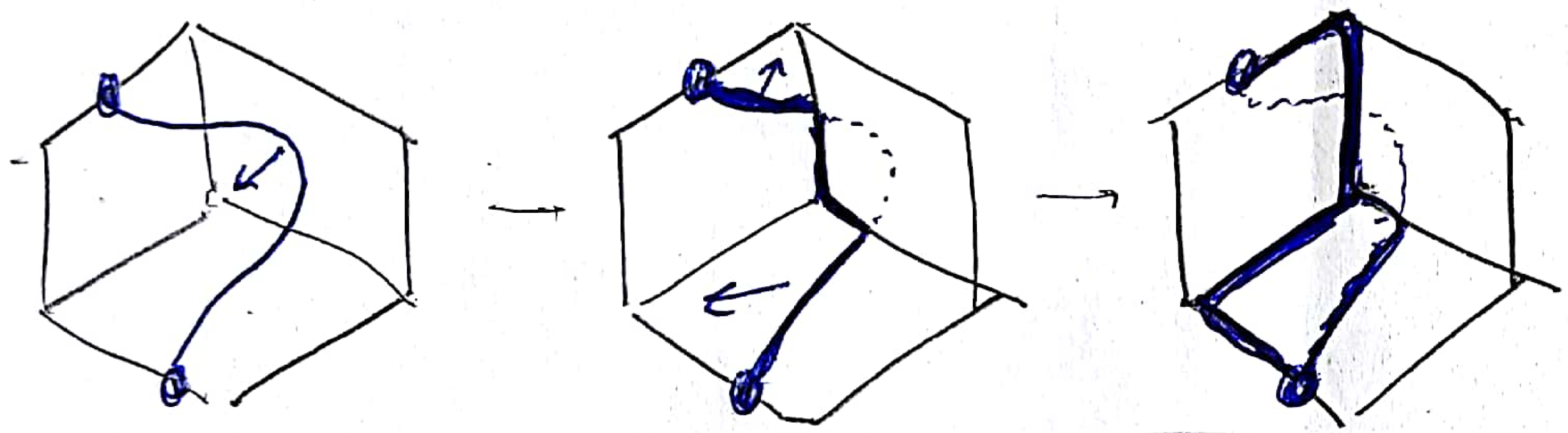
$d_C(x,y) = n$

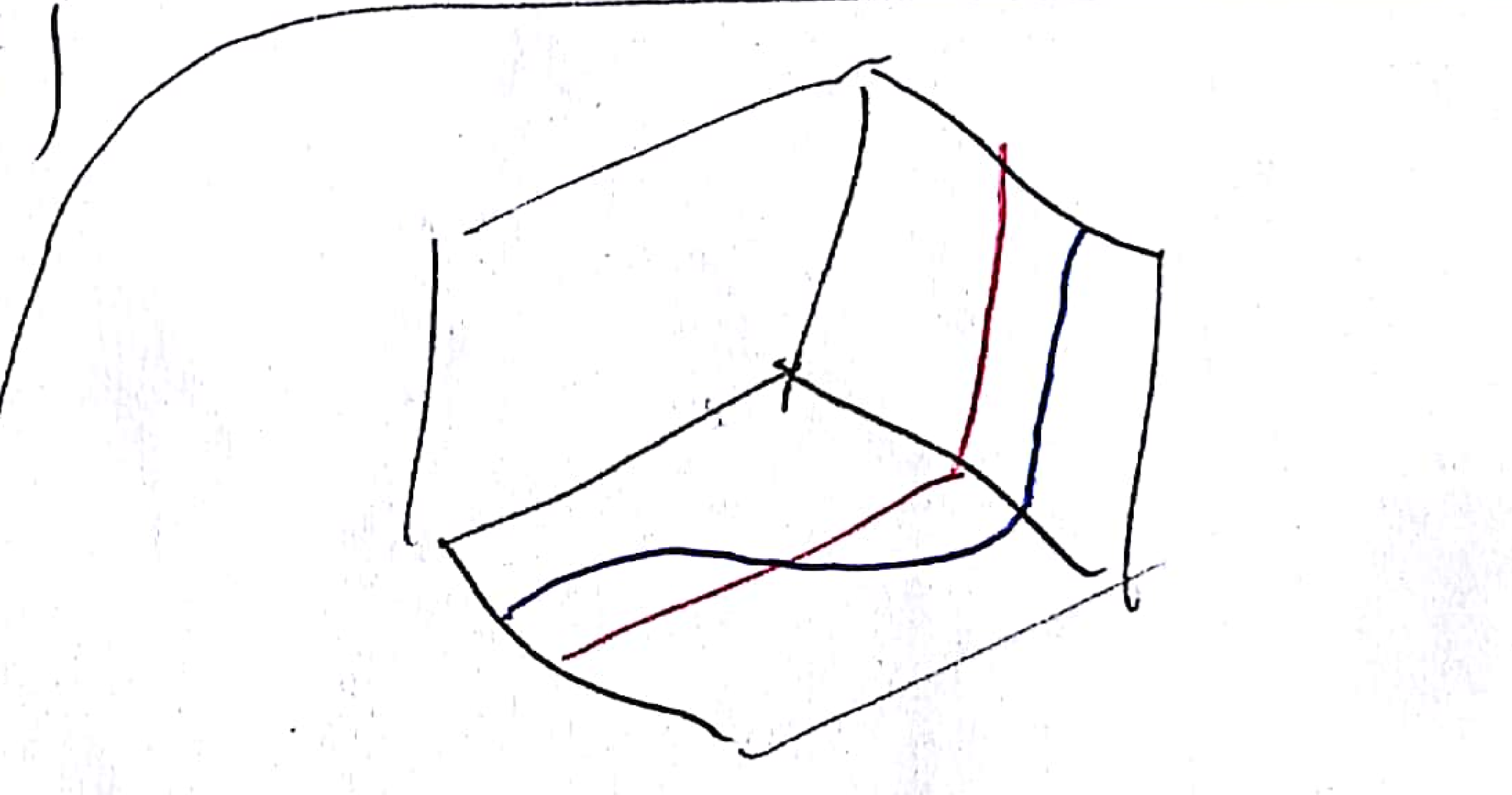
So each wall w joins opposite points, and each pair of walls crosses.

So $\text{Area}(M, W) \geq \text{Area}(M, W') = \frac{n(n-1)}{2}$.

Proof that continuous FAC (for self-reverse Finsler metrics) implies discrete FAC (for square-celled surfaces)

- Let M be a square-celled isometric filling of C_{2n} that has $m < \frac{n(n-1)}{2}$ cells. (counterexample to discrete FAC)
- Replace each cell by a unit square cut from l_1 plane.
- Each curve γ that connects boundary points can be pushed to the skeleton without increasing its length. (hence, γ is not a shortcut)





Unless γ avoids the vertices, and exits each square through the opposite side that it entered. (So γ runs parallel to a wall

- To fix this, we interrupt each of the n non-closed walls with a hemispheric "eye". Now every curve γ can be pushed to the skeleton without increasing its length, so the surface is an isometric filling
- It has $\mathrm{Len}(\partial M) = 2n$ and $\mathrm{Area}_{HT}(M) = 4m + 2n < 2n^2$. (piecewise-smooth counterexample to continuous FAC; can be smoothed)

Proof that discrete FAC implies cont FAC

From a continuous counterexample (M,F), we will construct a discrete counterexample $(M;W)$.

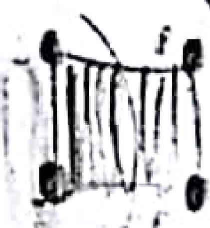
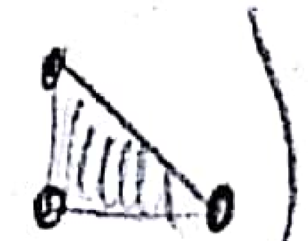
- Need to approximate Finsler metric with wallsystem!

Step 0: Approximate normed plane. $(\mathbb{R}^2, \|\cdot\|)$

Assume $\frac{1}{2}B^*$ is an integral polygon with vertices in $2\mathbb{Z}^2$.

- We will construct a periodic wallsystem W made of straight walls, such that

$d_W(x,y) = \|y-x\|$ whenever $x,y \in \mathbb{Z}^2$, and such that

$4\cdot \mathrm{Area}_W$ (unit square ) $= |B^*|$ (can also ensure that Area() $= \frac{1}{2}|B^*|$ if vertices in $4\mathbb{Z}^2$)

How?] - On the torus, $\mathbb{T}=\mathbb{R}^2/\mathbb{Z}^2$, the walls are closed curves. Its homotopy classes are integral vectors, because $\pi_1(\mathbb{T}^2)=\mathbb{Z}^2$.

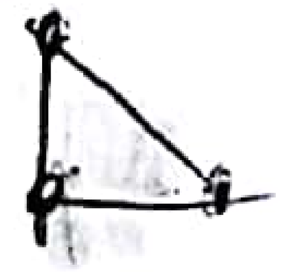
They are obtained from the ~~[illegible]~~ sides of the polygon B^* by the following moves:

- from each pair of equal opposite sides, discard one
- divide the remaining vectors by two.
- if not primitive, break them into primitive pieces $n\cdot v \longrightarrow n$ copies of v.
- rotate $\frac{1}{4}$ turn.

$\boxed{\text{This works! you check}}$

Proof that Finsler FAC implies discrete FAC | cont.

Let (M,F) be counterexample. Iso filling boundary of length $2L$. $Area_{HT}(M,F) < 2L^2$.

Triangulate: Break into small triangles, so that on each triangle T the metric F_x varies little from point to point. On each triangle. (Assume $T=$  $\subseteq \mathbb{R}^2$.)

Polyhedral approx: On each face, replace the norm F_x by the norm $\bar{F}_x = \sup_x F_x \to$ normed plane. On boundary sides, replace by infimum so $d_{\partial M}$ decreases, d_M increases } still isometric filling

~~Area increases little; still counterexample.~~

Also, choose small $\delta > 0$ and increase each $\bar{B}_x^*$ to make it a ~~...~~ polygon with vertices in ~~...~~ $\delta' \cdot \mathbb{Z}L^2$.

Eyes: ~~Break each side of the bo~~ Choose large $k \in \mathbb{N}$, and break each side into k pieces of equal length at points called portals. Insert a hemispheric eye. ~~...~~ Filling still isometric, still counterexample if k large enough.

Walls: Multiply metric by ~~...~~ $4\frac{\delta'}{k}$ so segments between portals are in $4\cdot \mathbb{Z}^2$.

Replace ~~...~~ metric on flat faces using lemma, and lengths of

Replace metric on ~~...~~ hemispheric eyes by PL A