One-Dimensional Computational Topology

II. Homotopy of Curves on Surfaces

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The question

Given two curves on an orientable surface, are they homotopic?

But first...

- What does "*orientable surface*" mean?

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- What does "***Given***... *an orientable surface*" mean?

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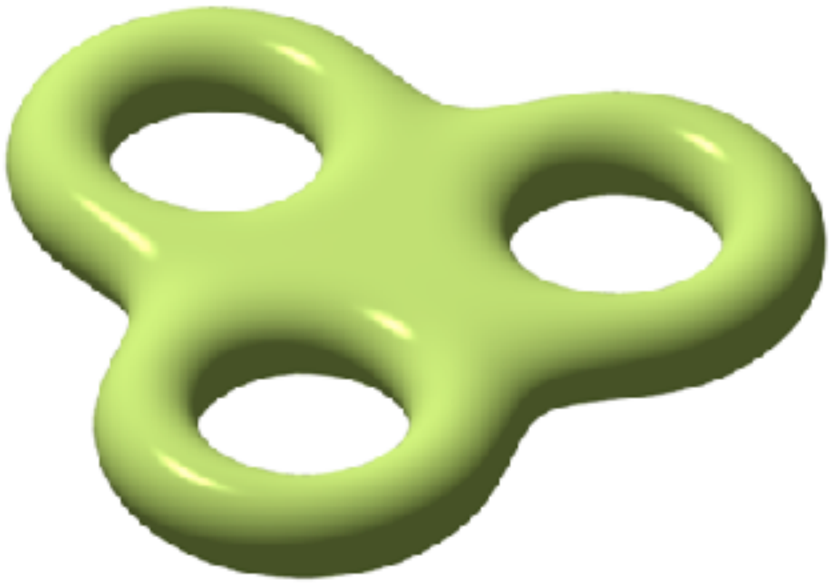
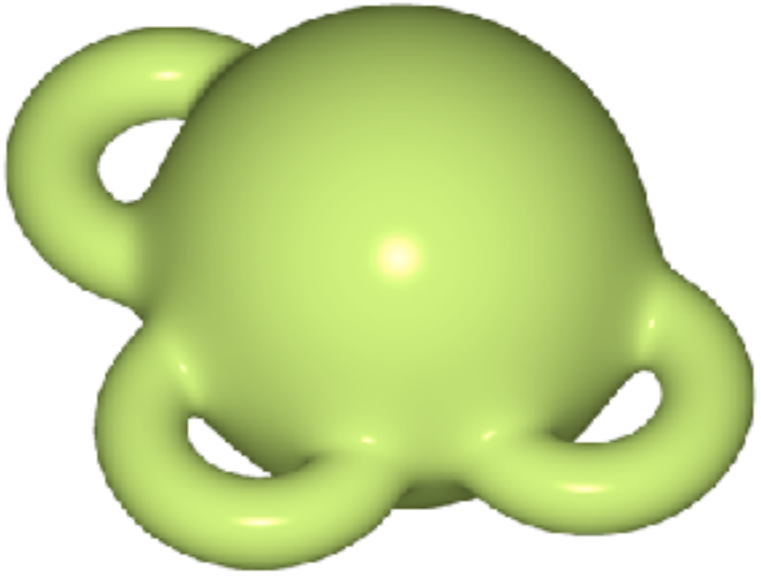
- What does "*orientable surface*" mean?
- What does "***Given****... an orientable surface*" mean?
- What does "***Given*** *two curves*" mean?
- What does "*homotopic*" mean?

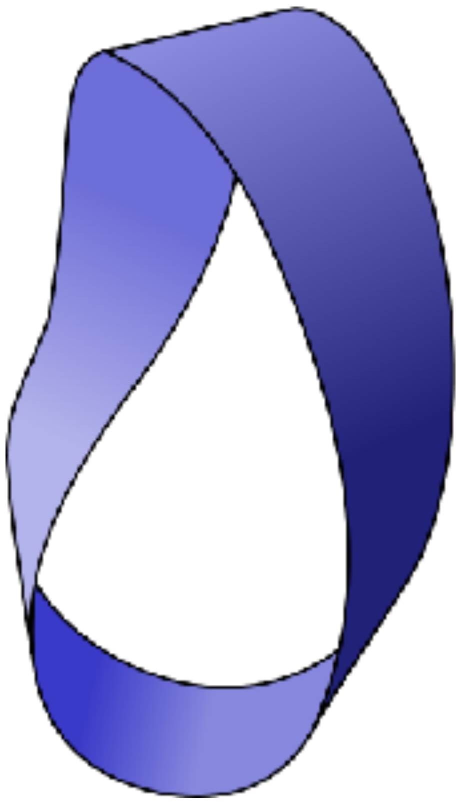
Surface maps

Surfaces = 2-manifolds

- Connected, compact, Hausdorff space in which every point has a neighborhood homeomorphic to the plane.
- An *orientable* surface does not contain a Möbius band.

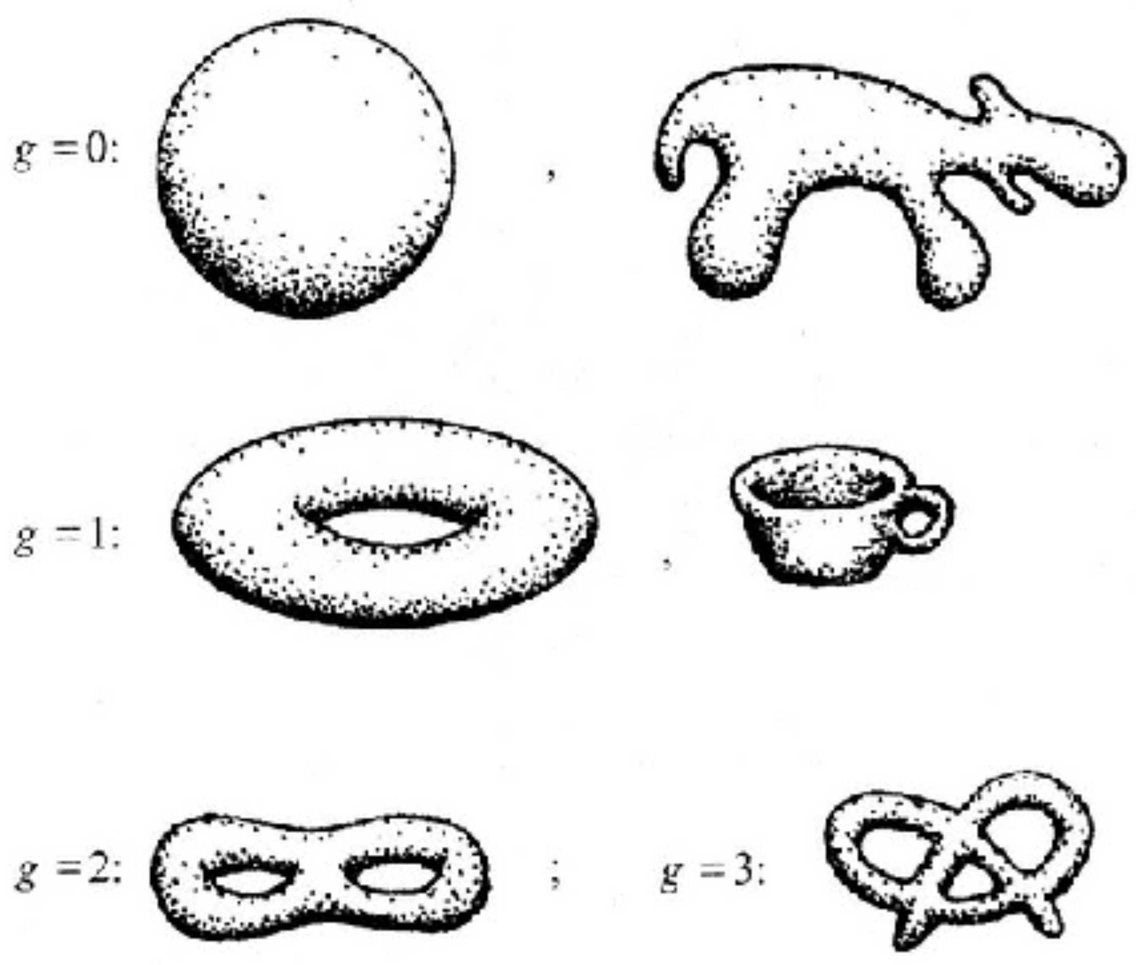






Surface classification

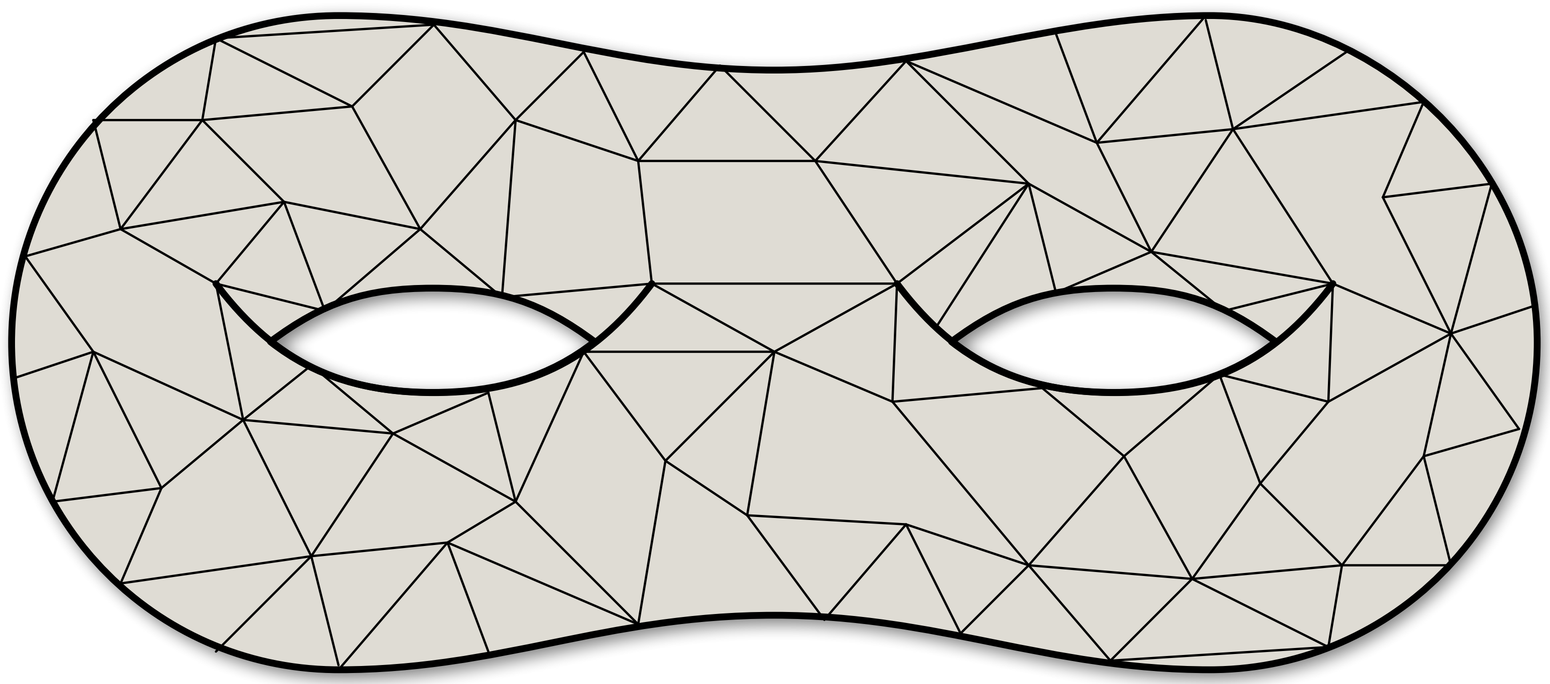
Every orientable surface is homeomorphic to a sphere with g handles, for some integer $g \geq 0$, called its *genus*.



Roger Penrose, The Road to Reality *(2004)*

Surface map

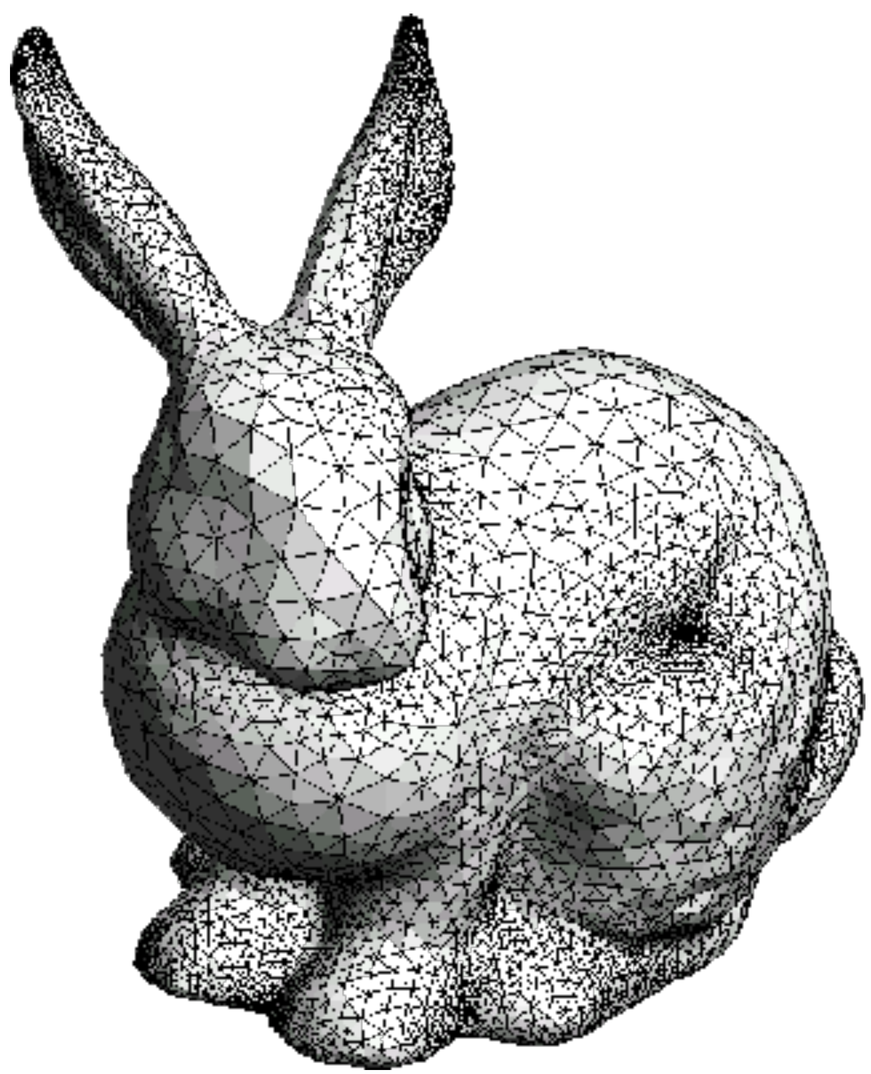
- Graph embedded on a surface so that each face is a disk
- Equivalently: Polygons glued together into a closed surface

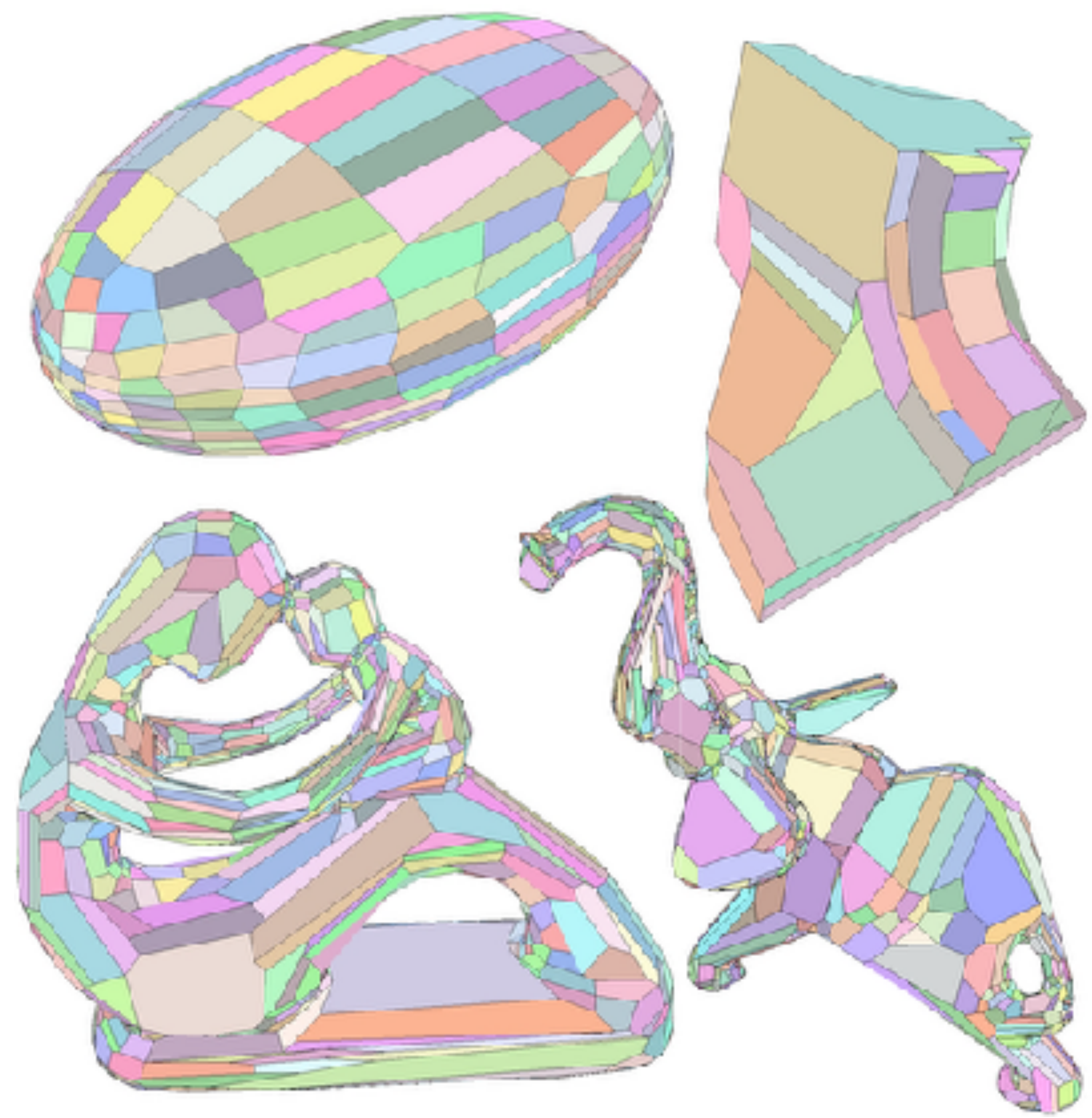


[Riemann 1857; Heawood 1890; Poincaré 1895; Heffter 1898; Dehn Heegaard 1907; Kerékjártó 1923; Radó 1937; Edmonds 1960; Youngs 1963; Gross Tucker 1987 ; Mohar Thomassen 2001; ....]

Surface map

- The standard surface representation in graphics and geometric modeling....

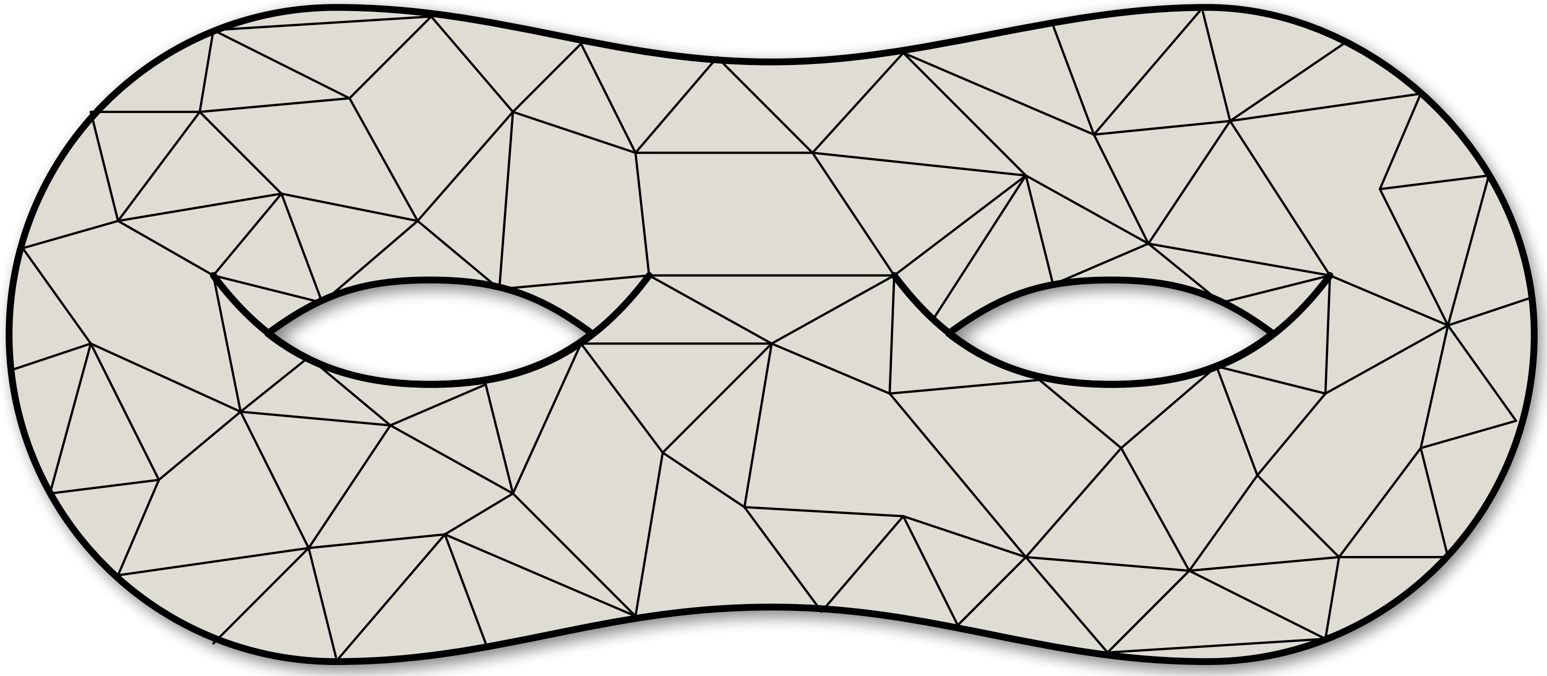




[Pellenard Morvan Alliez '12]

Surface map

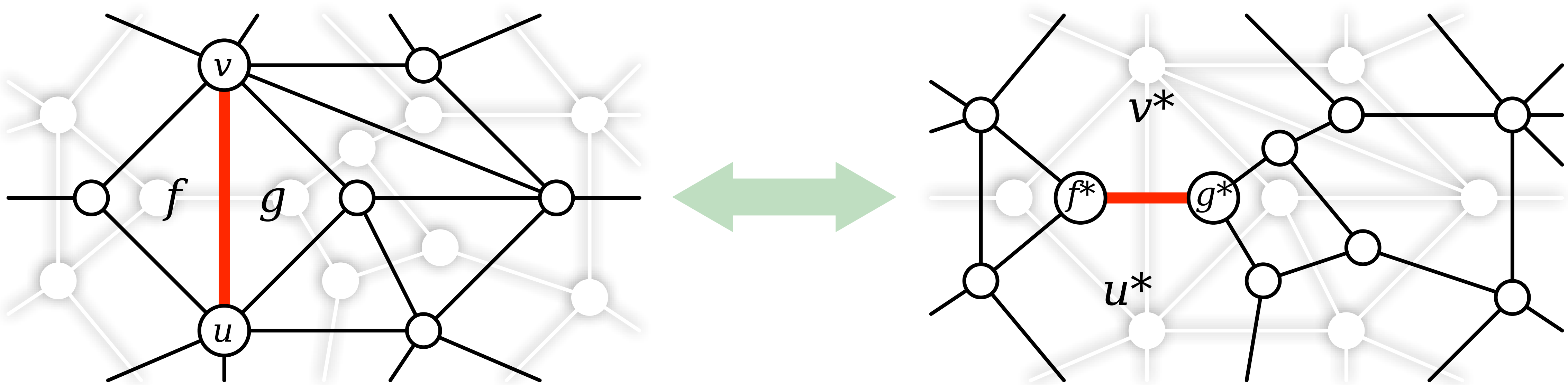
- The standard surface representation in graphics and geometric modeling, *but without vertex coordinates*



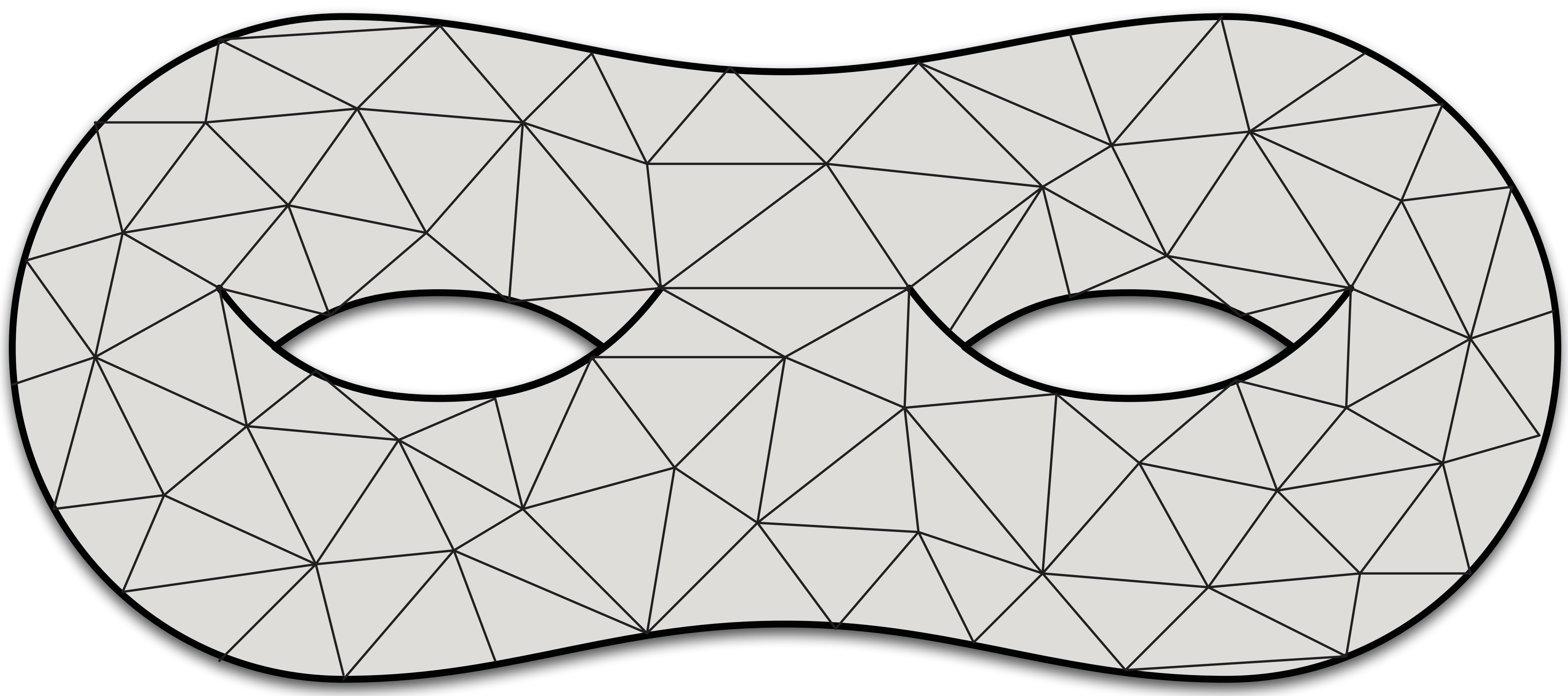
Map duality

Every surface map $\Sigma = (V, E, F)$ has a natural *dual* map $\Sigma^* = (F^*, E^*, V^*)$ on the same surface:

- vertices of Σ^* = faces of Σ
- edges of Σ^* = edges of Σ
- faces of Σ^* = vertices of Σ



Map duality



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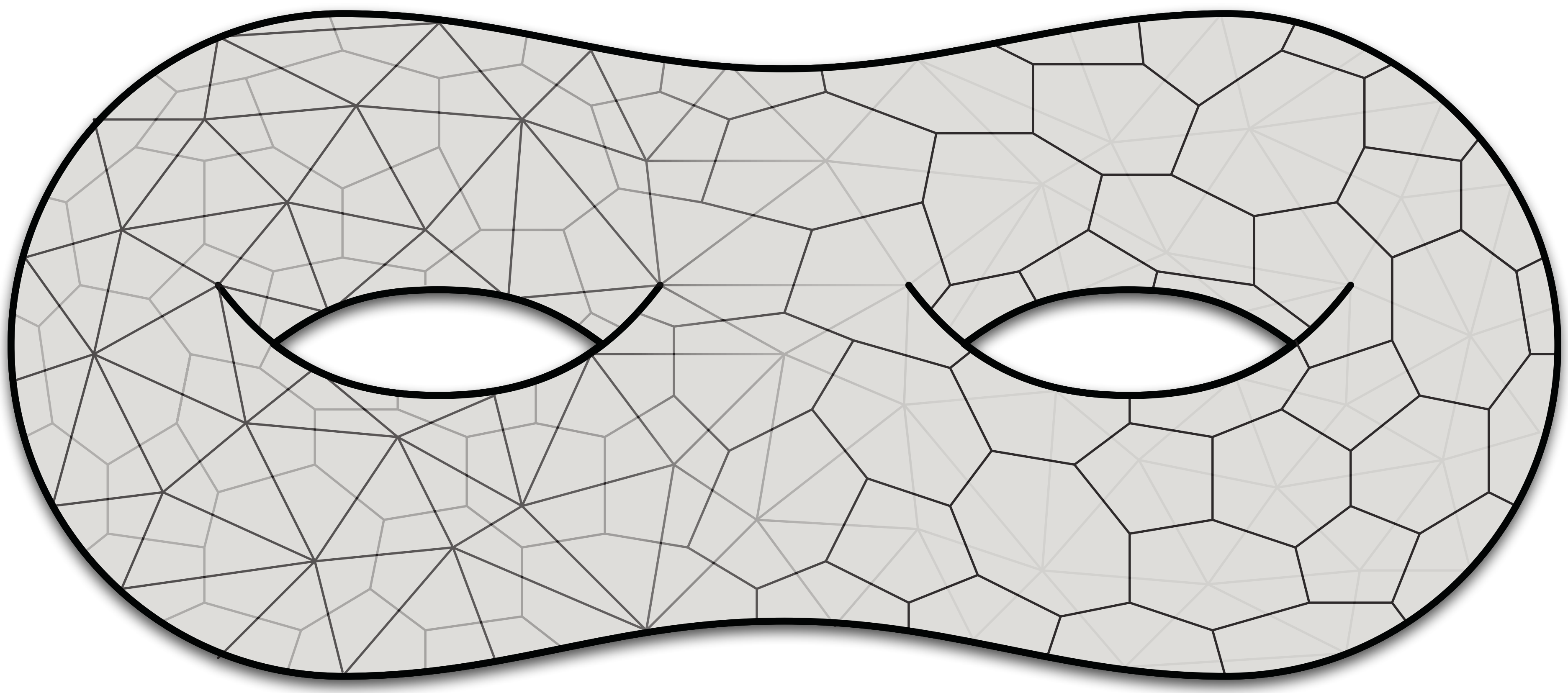

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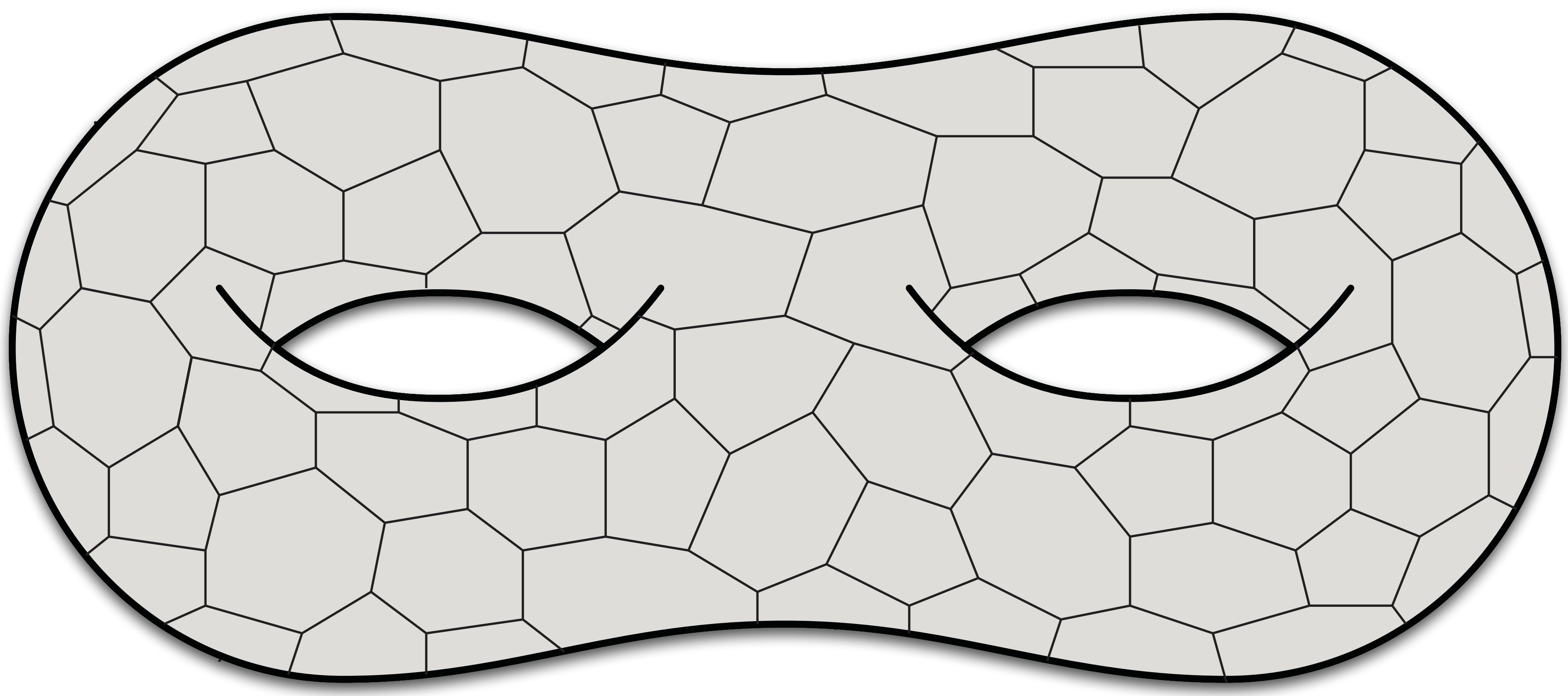
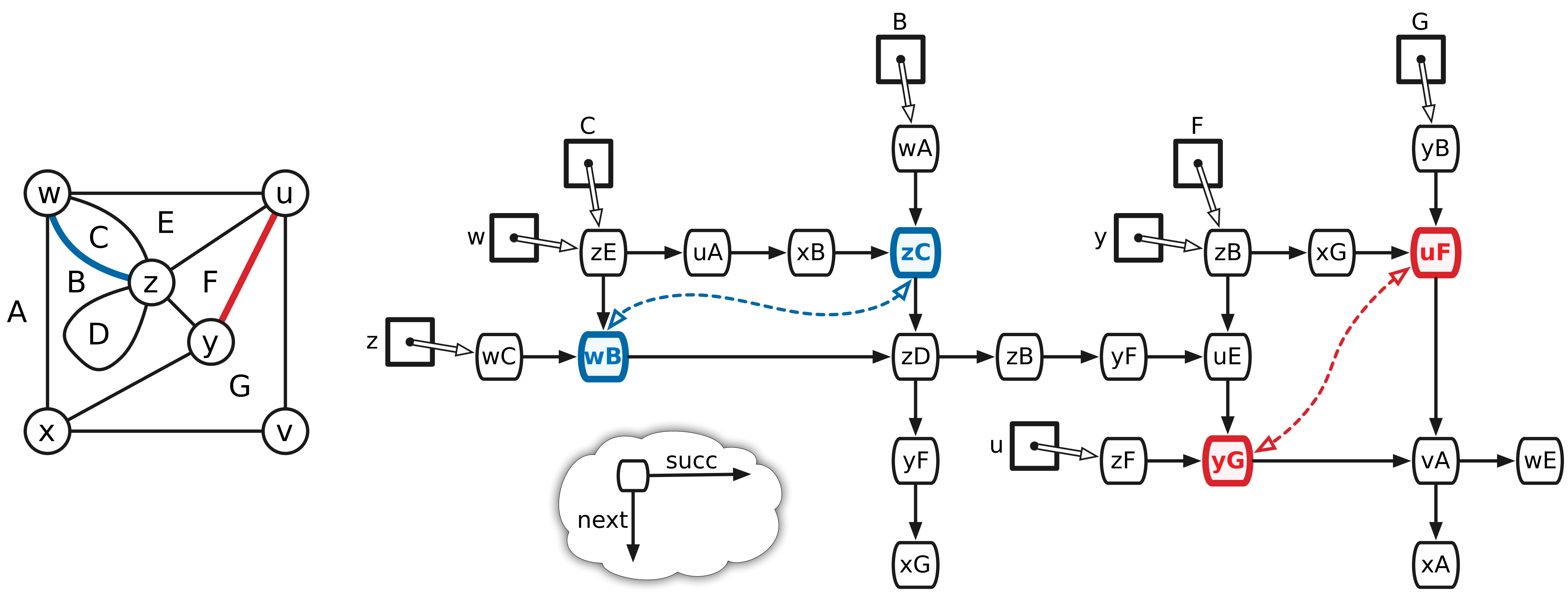


Data structure

- Both rotation system of Σ (listing edges around each vertex) and rotation system of Σ* (listing edges around each face).
- The *same* data structure represents *both* Σ and Σ*.



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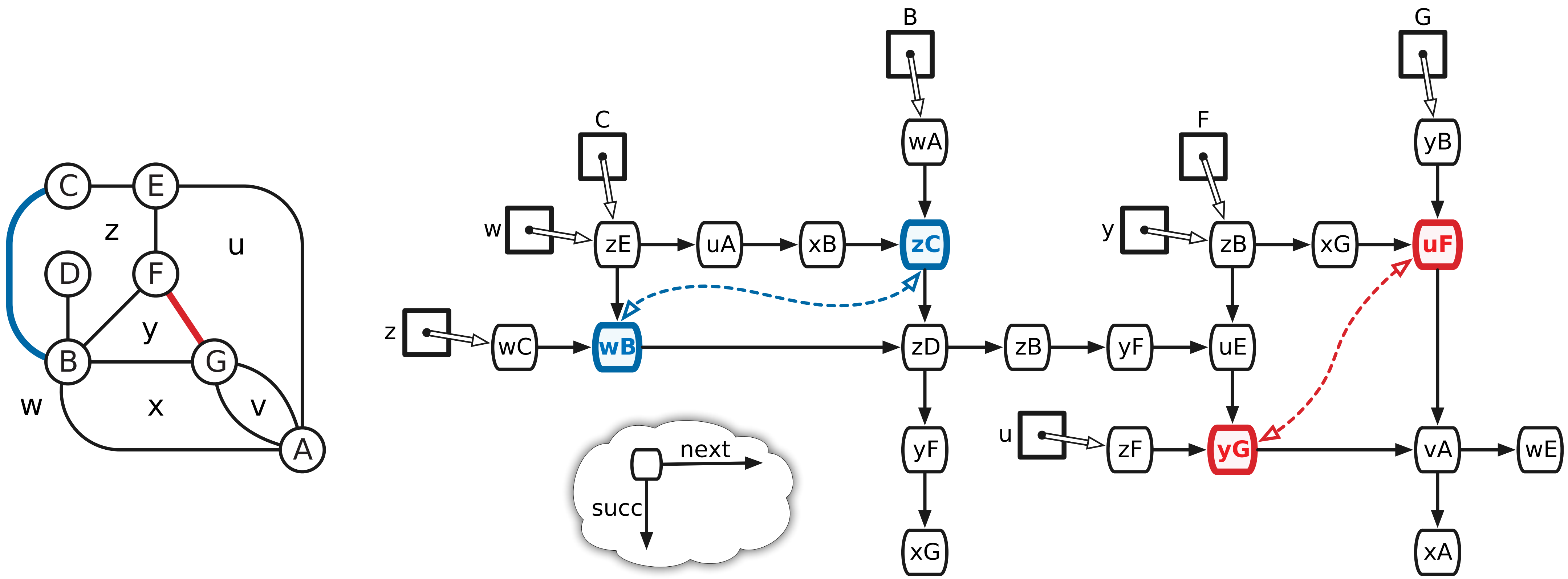


Data structure

[Hamilton 1856] [Kirkman 1856] [Cayley 1857] [Heffter 1891] **[Brückner 1900]**....

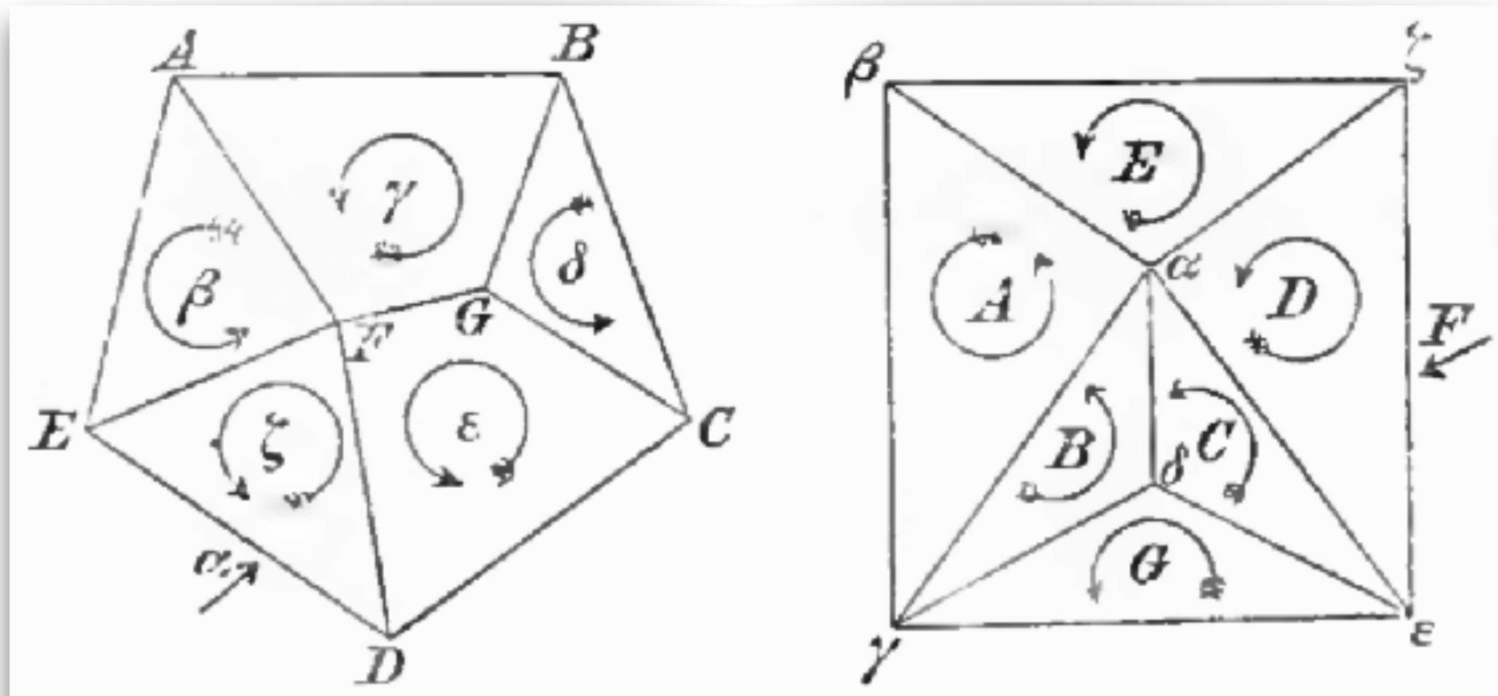


Fig 52a Fig 52b.

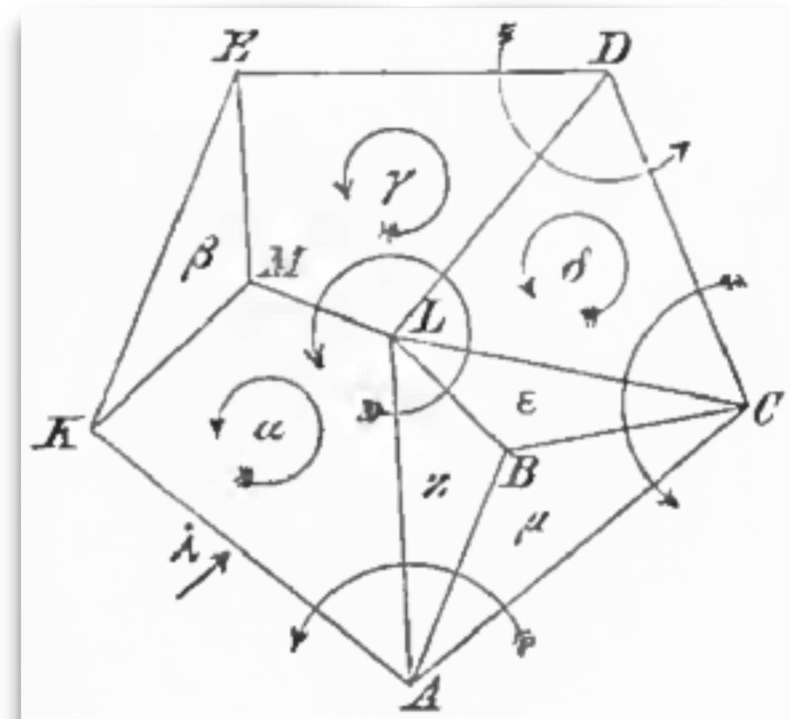


Fig. 53.

$$\alpha = ALMK, \quad \beta = KME, \quad \gamma = MLDE, \quad \delta = LCD, \quad \varepsilon = BCL,$$
$$A = \alpha\lambda\mu\varkappa, \quad B = \varkappa\mu\varepsilon, \quad C = \mu\lambda\delta\varepsilon, \quad D = \lambda\gamma\delta, \quad E = \beta\gamma\lambda,$$
$$\varkappa = ABL, \quad \lambda = KEDCA, \quad \mu = ACB.$$
$$K = \alpha\beta\lambda, \quad L = \varkappa\varepsilon\delta\gamma\alpha, \quad M = \alpha\gamma\beta.$$

Euler's formula

- For *every* map (V, E, F) on the orientable surface of genus g:

$$\boxed{V - E + F = 2 - 2g}$$

- $\chi := 2-2g$ is the *Euler characteristic* of the map / surface.

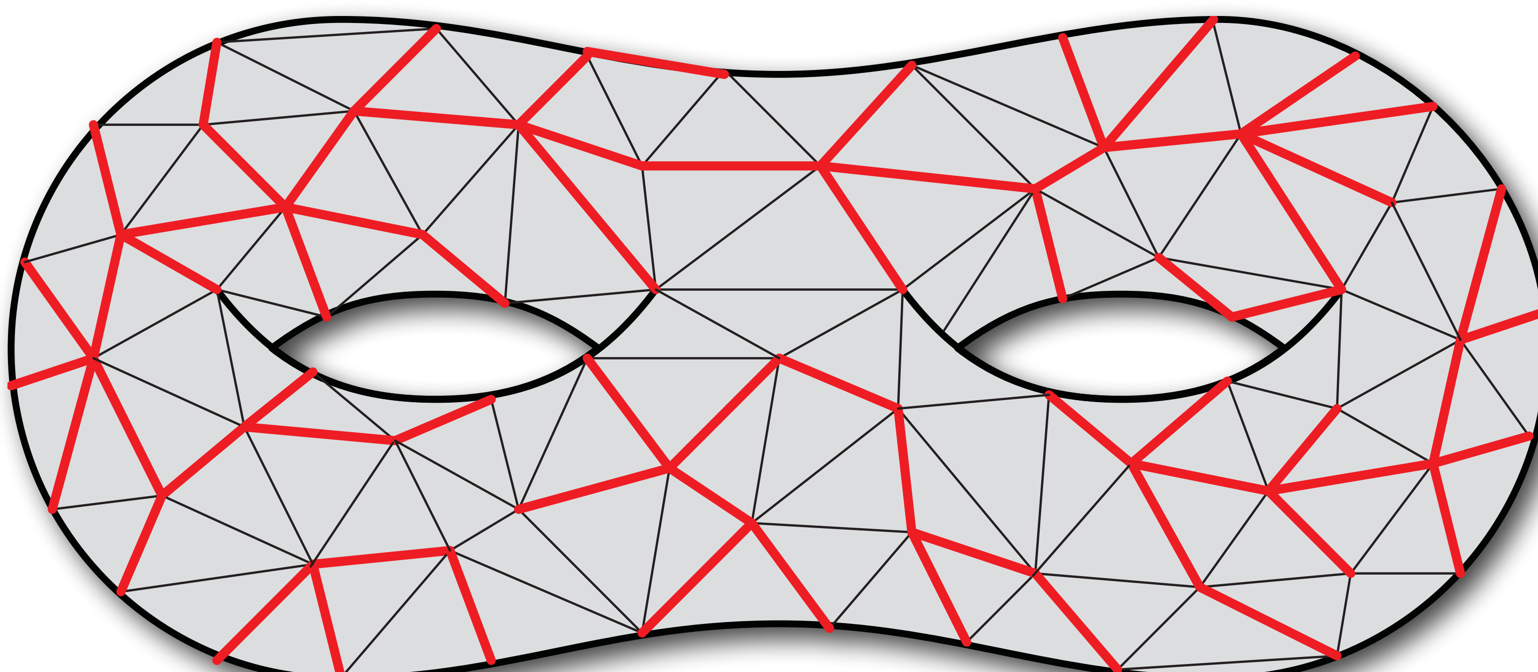
[Descartes c.1630 (via Leibniz 1676 (via Foucher de Careil 1859)),
***Euler 1750**, ~~Euler 1753~~, ~~Karsten 1768~~, ~~Meister 1784~~,*
Legendre 1794, Hirsch 1807, l'Huillier 1811, ~~Cauchy 1811~~,
~~Grunert 1827~~, Von Staudt 1847, Cayley 1861, ~~Listing 1861~~, ...]

Tree-cotree decomposition

[von Staudt 1847] [Dehn 1936] [Biggs 1971] [Eppstein 2003]

A *partition* of the edges into three disjoint subsets:

- A spanning tree ***T***
- A spanning cotree ***C*** — C^* is a spanning tree of G^*
- Leftover edges ***L*** := $E \setminus (C \cup T)$ — Euler's formula implies $|L| = 2g$



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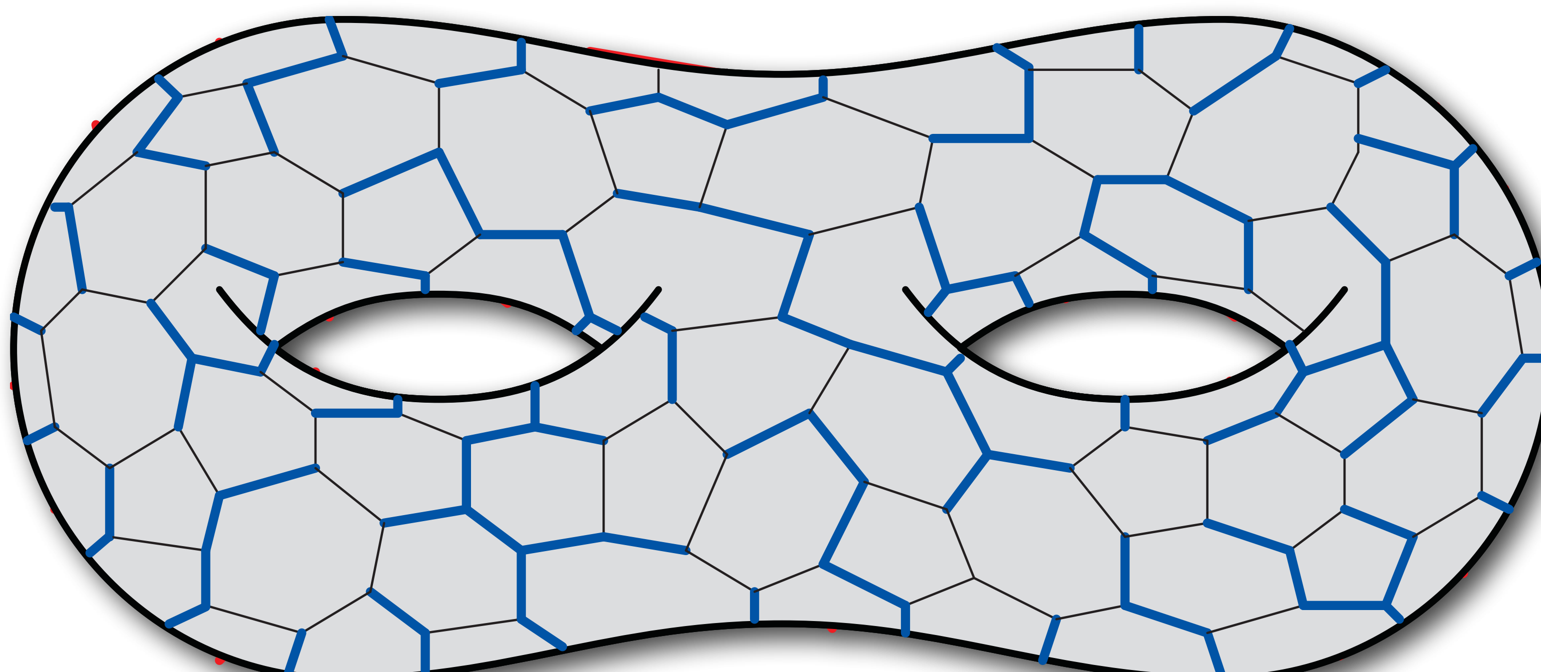

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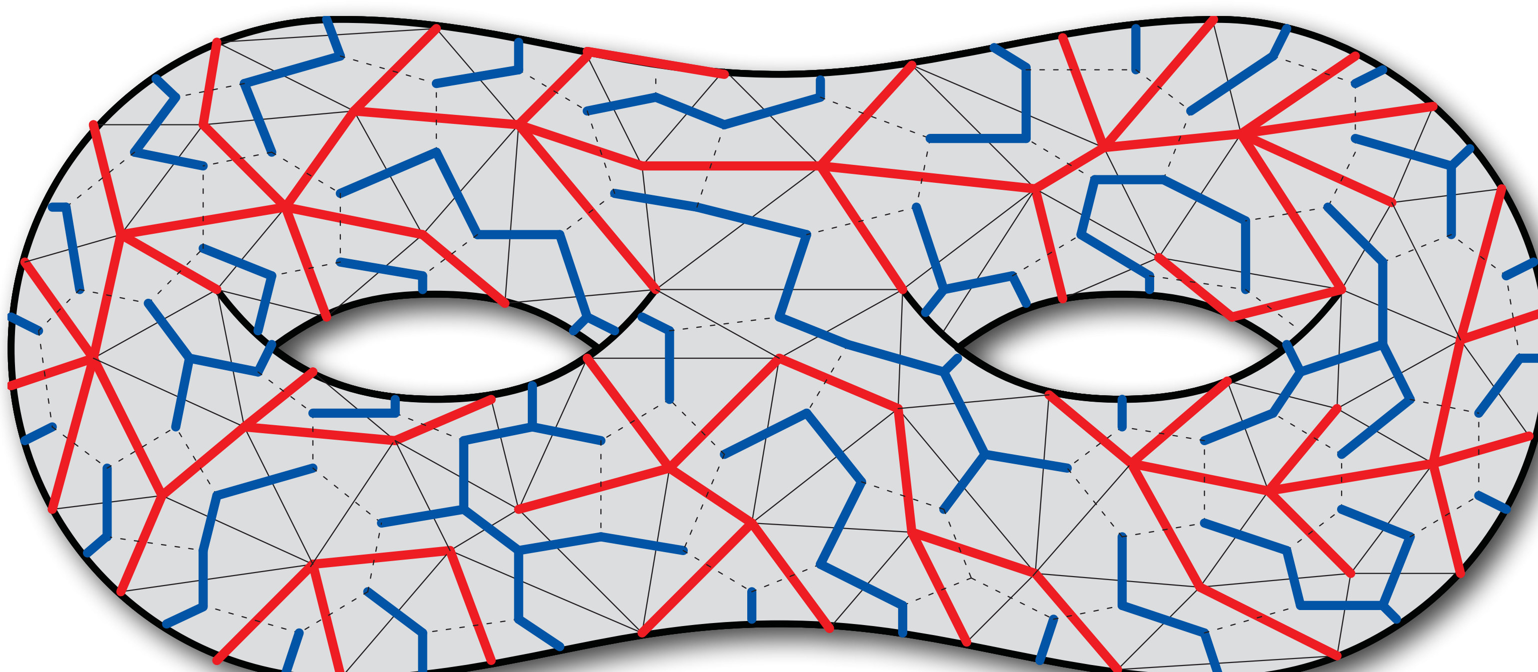

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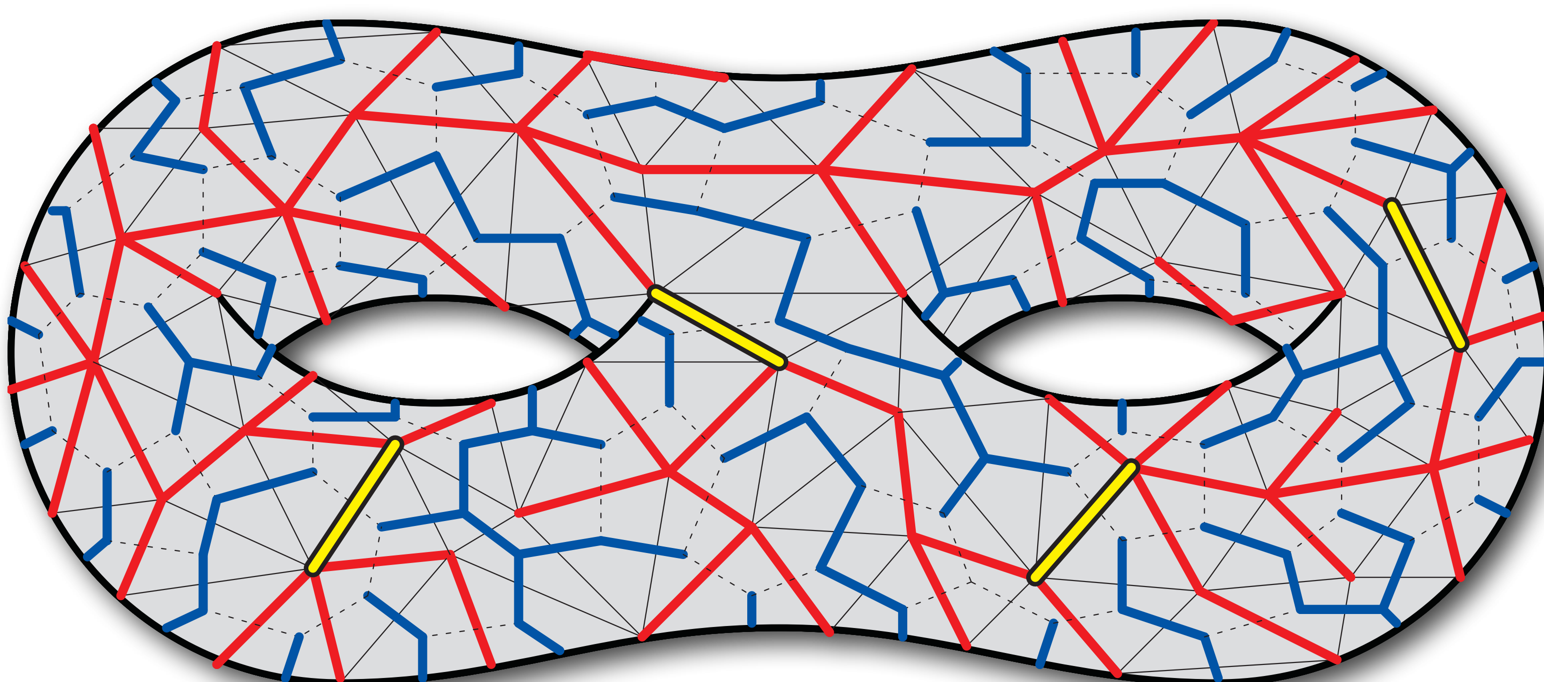
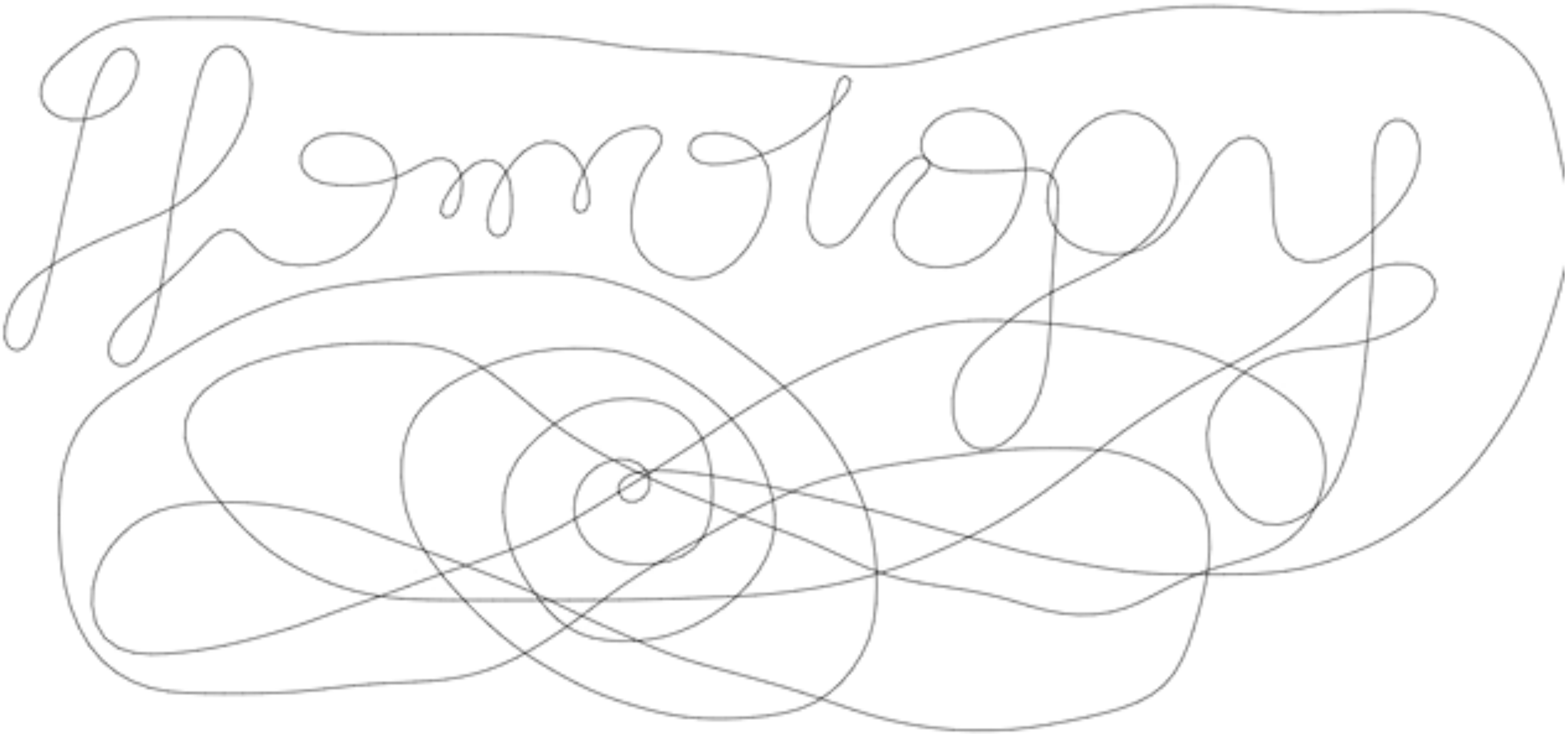


Homotopy



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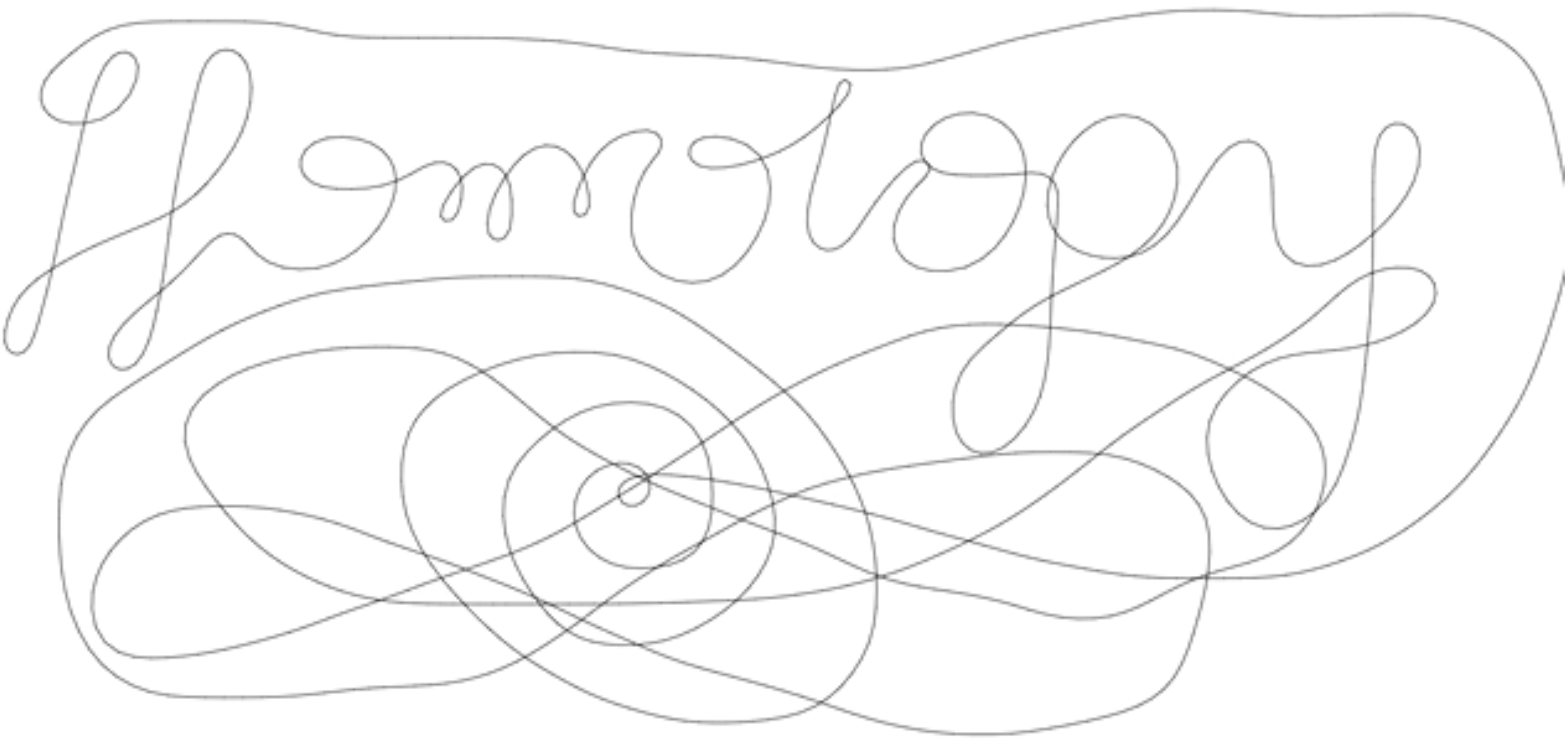
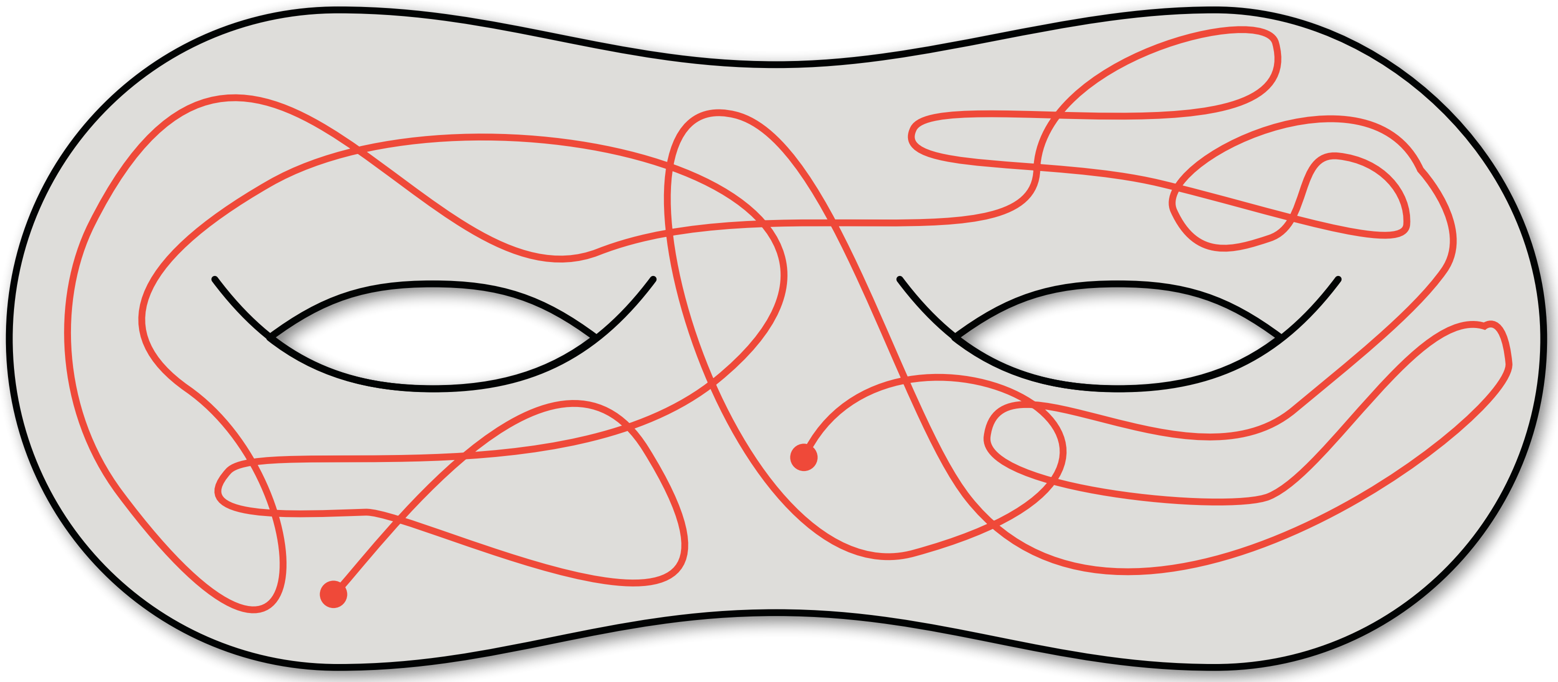


Homotopy

- Continuous deformation



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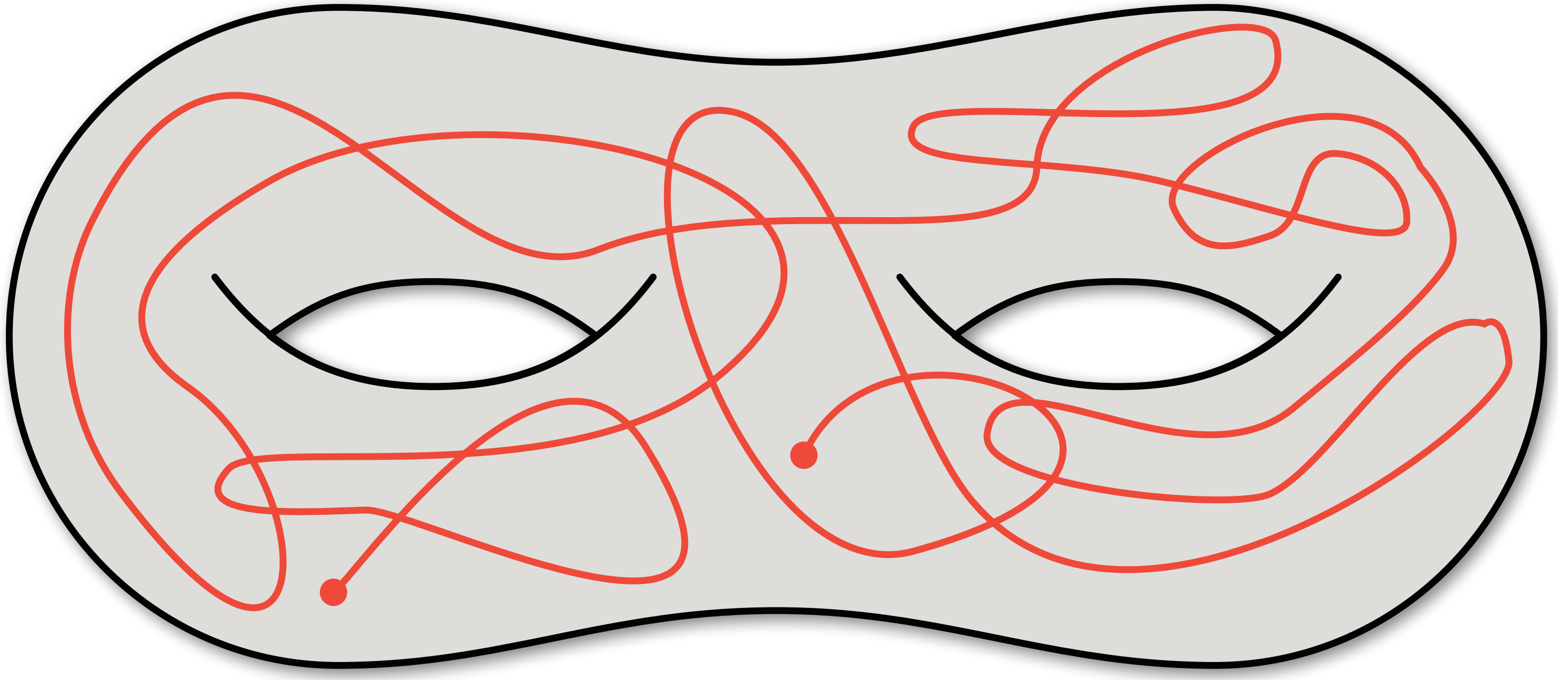

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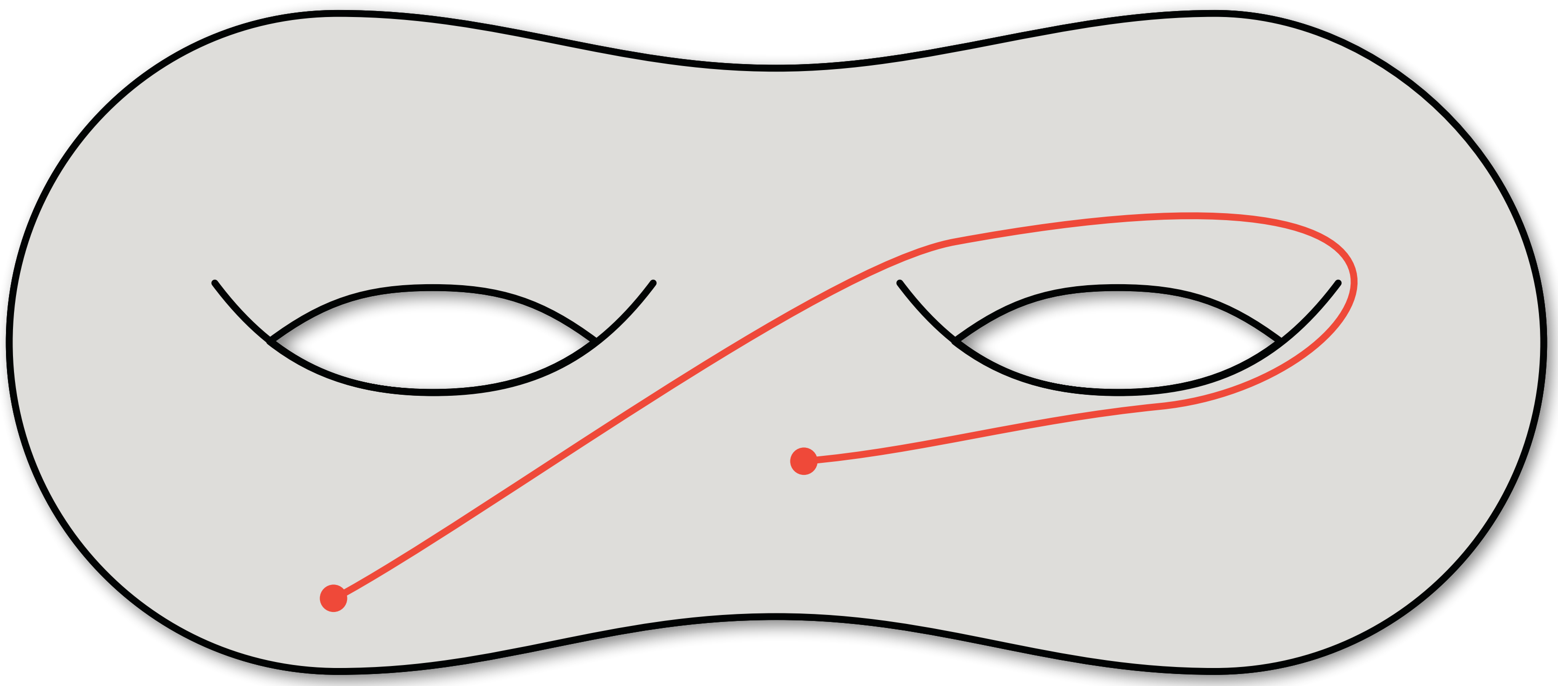
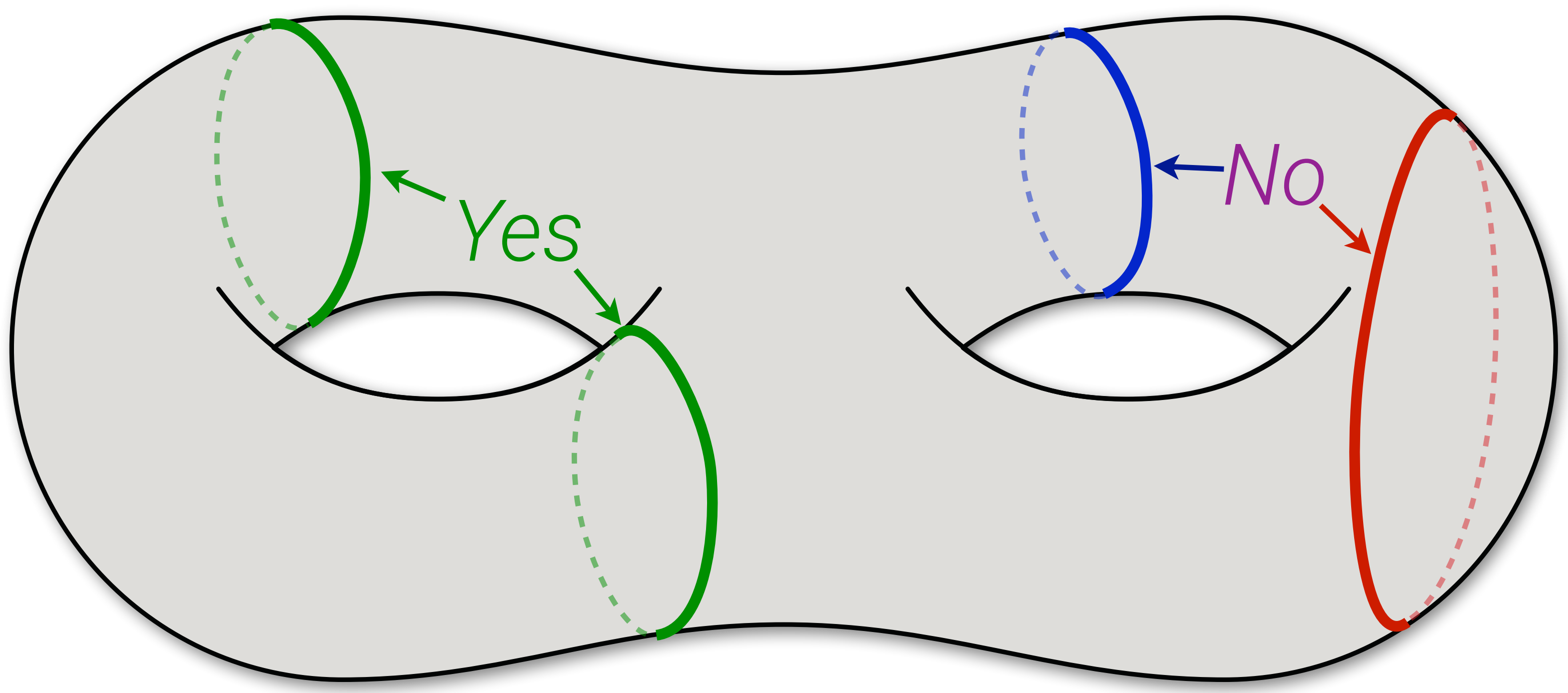


The homotopy problem

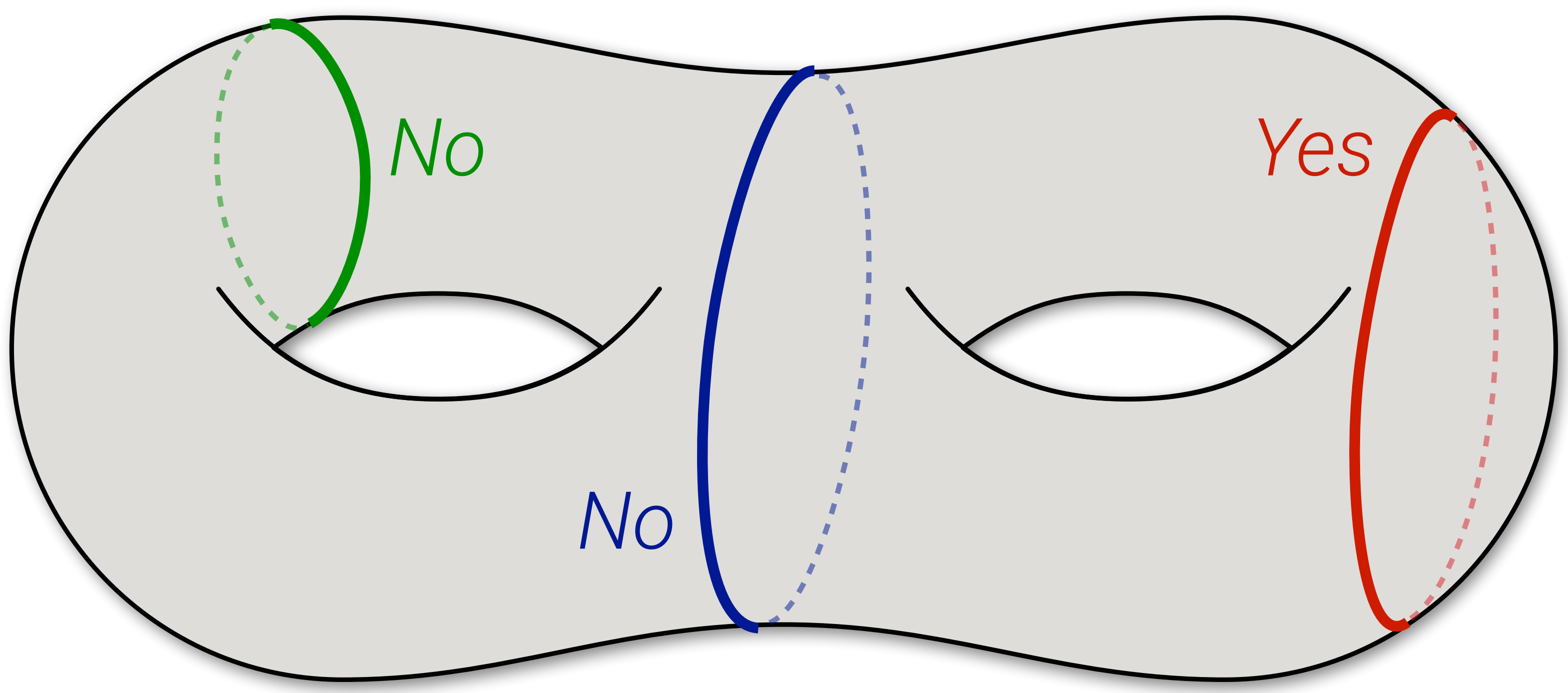
- Given two paths or cycles α and β in an orientable surface, can α be continuously deformed into β?



- ... are α and β *homotopic* / in the same *homotopy class*?

The contractibility problem

- Given a cycle α in an orientable surface, can α be continuously deformed to a point?



- ...is α *contractible*?

Max Dehn 1912

Transformation der Kurven auf zweiseitigen Flächen.

Von

M. Dehn in Kiel.

Das Problem, das uns im folgenden beschäftigen wird, ist eines der einfachsten der Topologie: *Gegeben sind zwei geschlossene Kurven auf einer geschlossenen zweiseitigen Fläche, es ist zu untersuchen, ob sie durch stetige Deformation ineinander übergeführt, „ineinander transformiert" werden können.* Die Lösung des Problems für Flächen mit einem Geschlecht $p > 1$ mit Hilfe der „Polygongruppen" und demgemäß auf Grund der Metrik der hyperbolischen Ebene ist naheliegend und z. B. von Poincaré (Rend. Circ. Mat. Pal. 1905) angedeutet, von mir in der Arbeit Math. Ann. 71 ganz genau entwickelt.*) In derselben Arbeit habe ich auch eine Methode angegeben, um ohne Hilfe der Metrik rein topologisch die Frage zu entscheiden. Bei der *Begründung* dieser Methode habe ich aber sehr wesentlich Eigenschaften von Figuren der hyperbolischen Ebene benutzt. — Für Flächen vom Geschlecht $p = 0$ und $p = 1$ ist die Lösung des Problems sehr einfach: im ersten Fall sind alle Kurven ineinander transformierbar, im

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Transformation of curves on two-sided surfaces

by

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The problem we shall consider in what follows is one of the simplest in topology: ***Given two closed curves on a closed two-sided surface, decide whether whether they can be "transformed into each other" by continuous deformation.***

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Ancient history

- Solvable via covering spaces *[Schwarz 1870s, Poincaré 1905]*
- Algorithms using hyperbolic geometry *[Dehn 1911]*

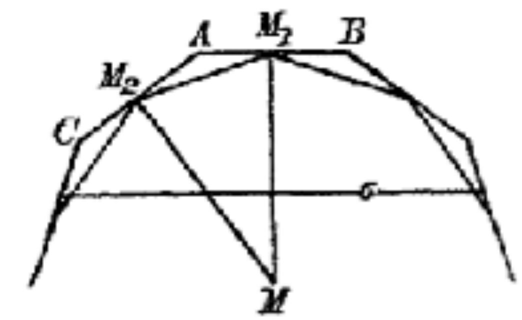
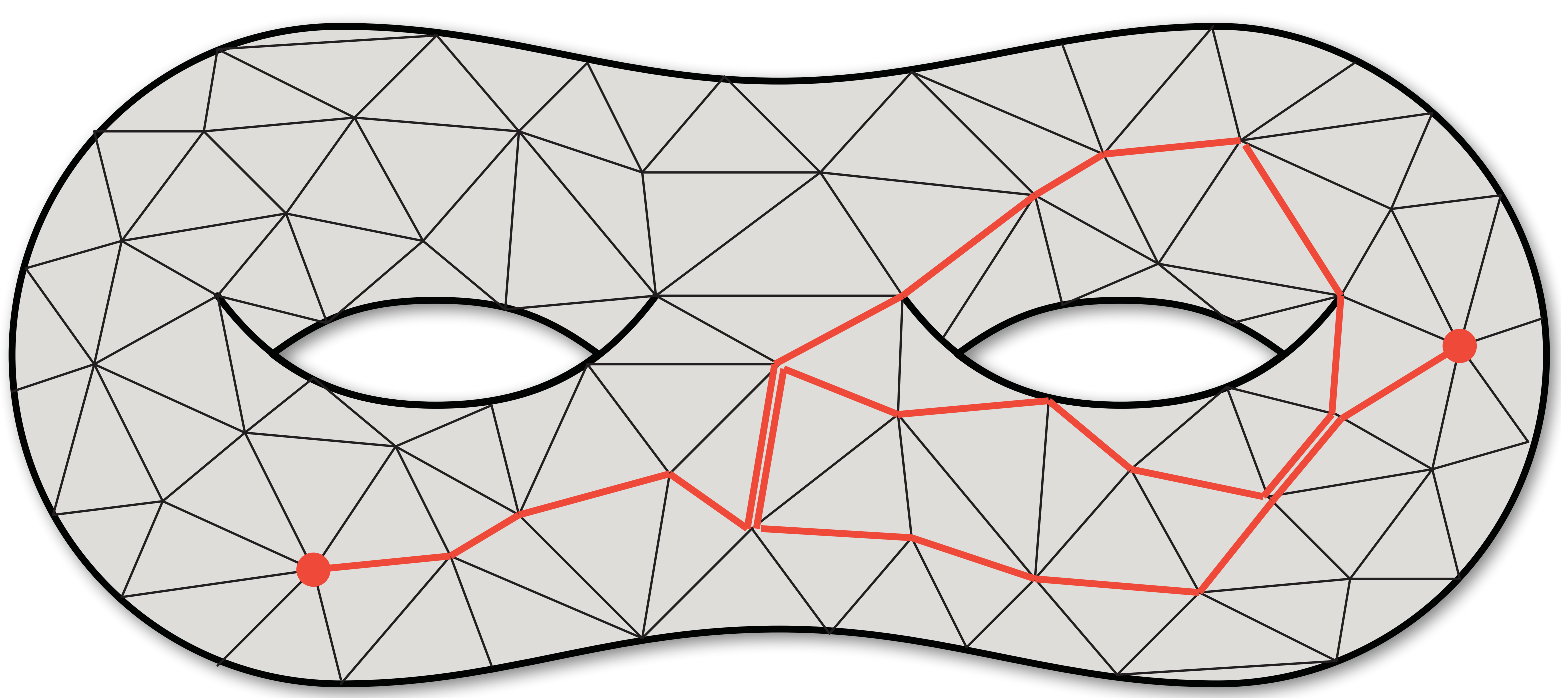


Fig. 2.

- **Dehn 1912: Purely symbolic/combinatorial algorithms**
  - Dehn's technique now called *small cancellation* *[Lyndon Schupp 1977, Epstein et al 1992, McCammond Wise 2000]*
  - For any *fixed* surface, Dehn's algorithm runs in $O(\ell)$ time, where ℓ = length of input curve(s).

Input

- Surface map Σ with complexity n and genus g
- A closed walk of length ℓ in the graph of Σ



Input

This is what "given a surface" means!

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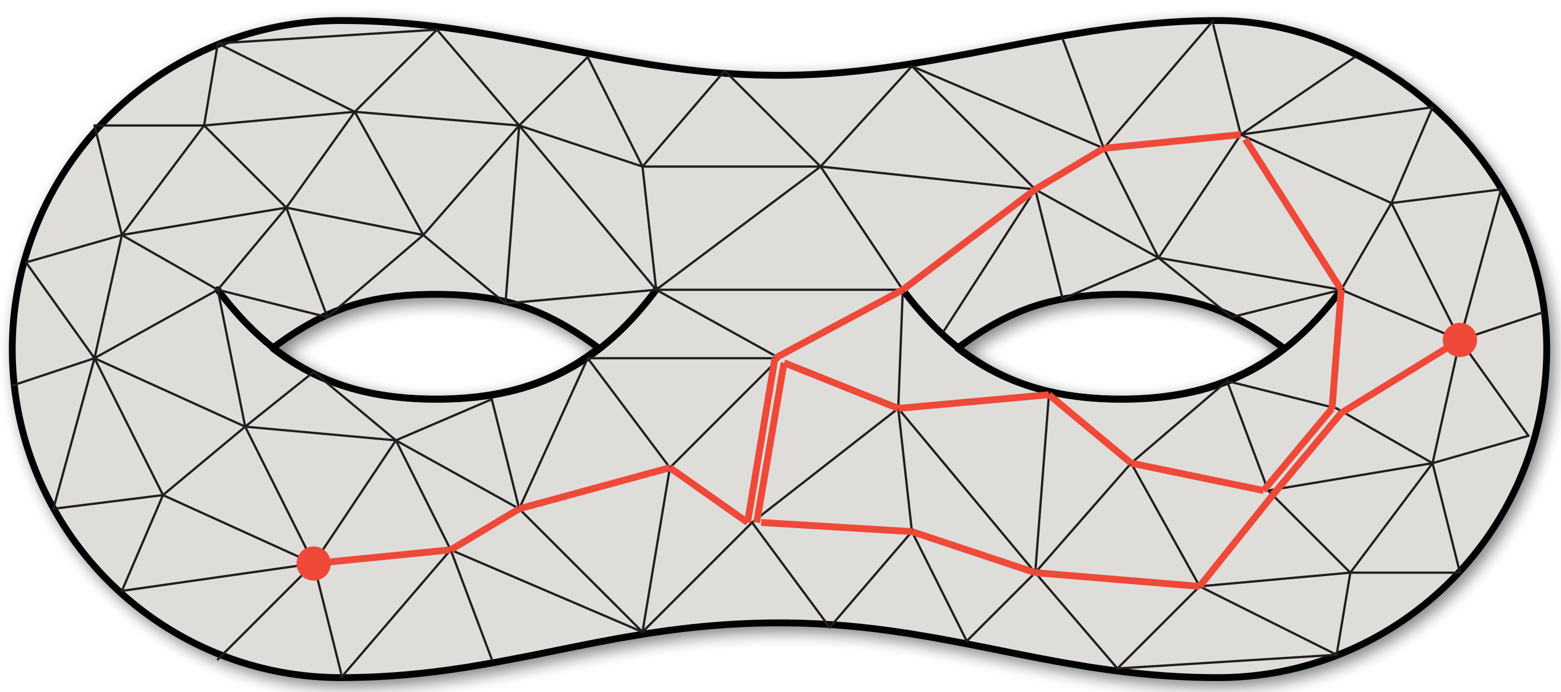

Input

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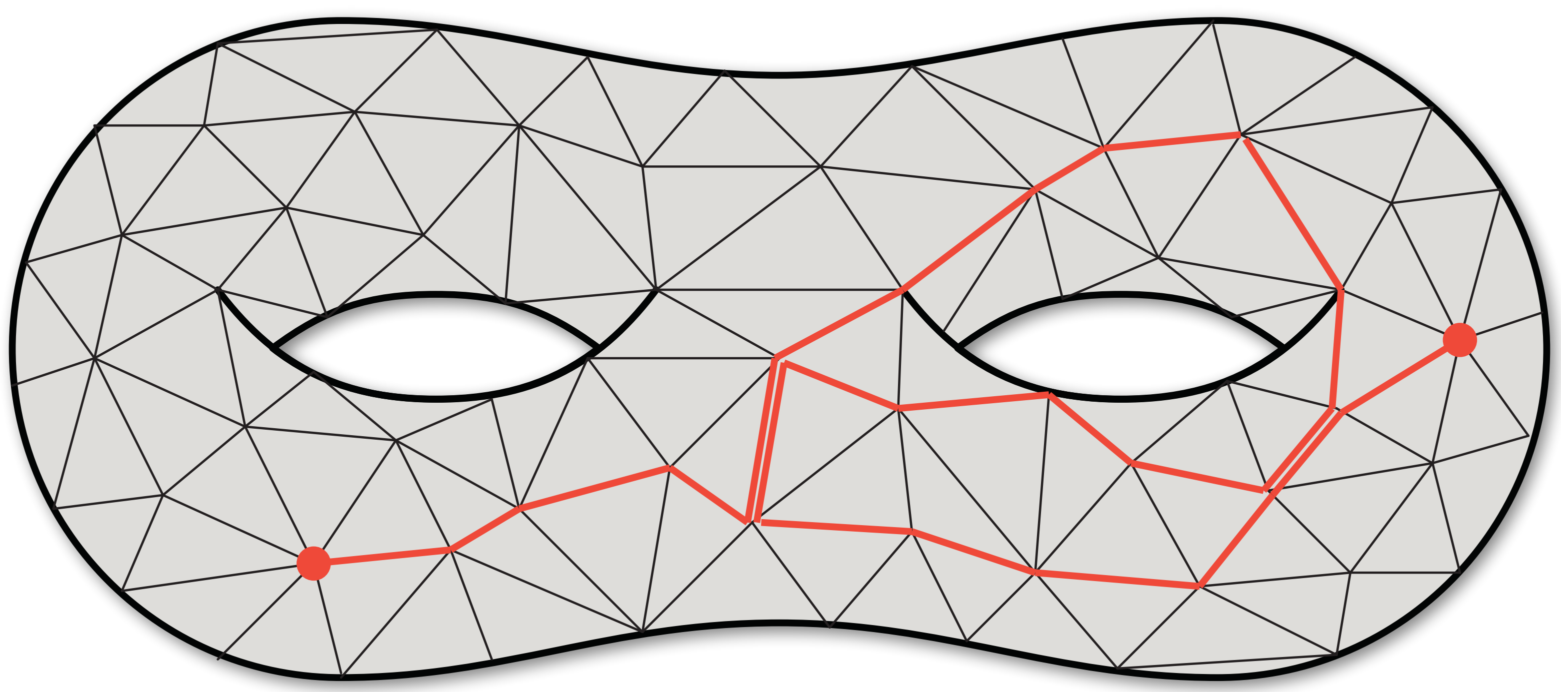
This is what "given a surface" means!

- A closed walk of length ℓ in the graph of Σ

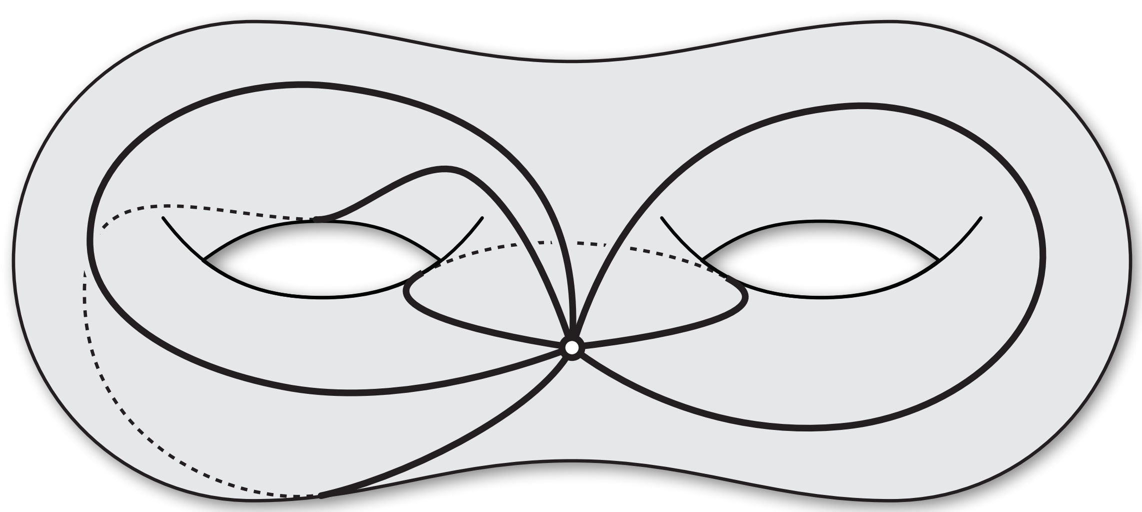
This is what "given a curve" means!



Dehn’s algorithm

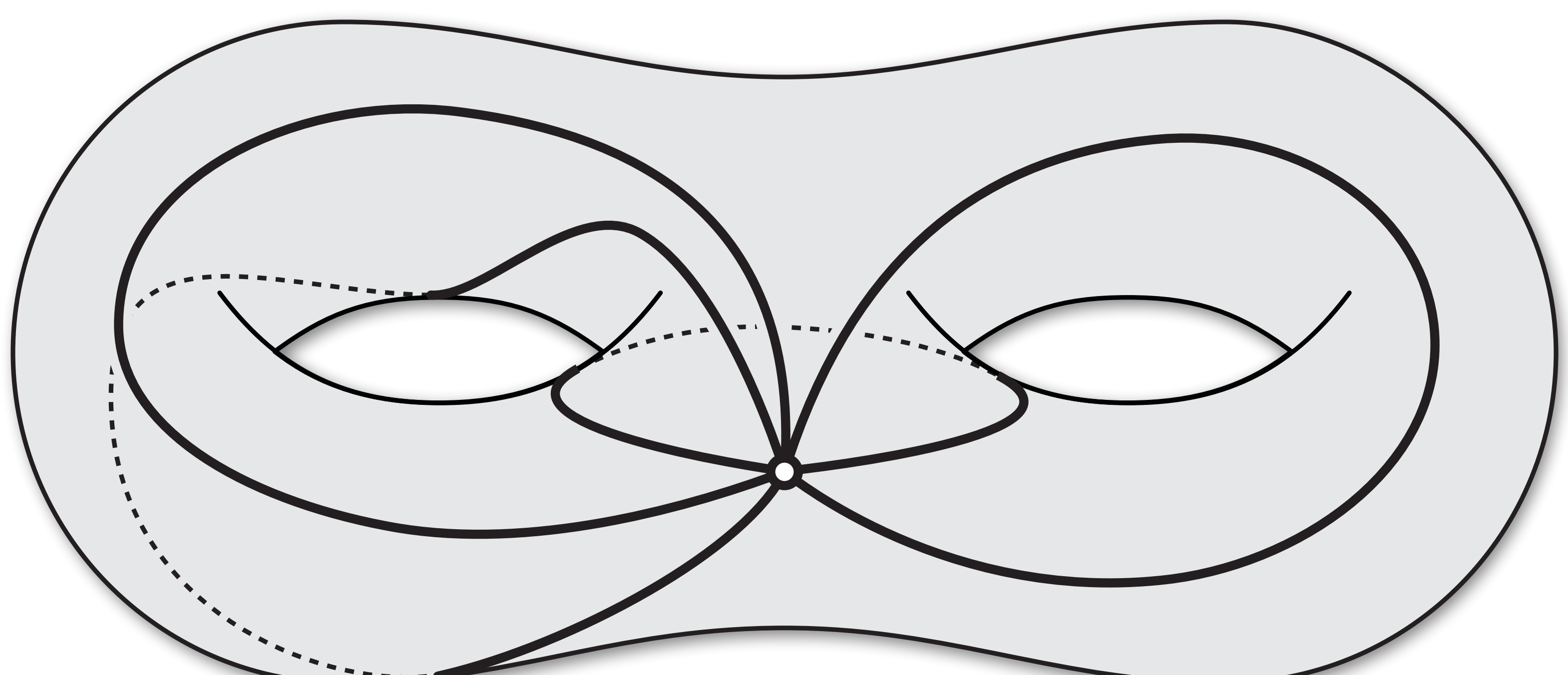
System of loops

- *2g* loops with common basepoint, which cut the surface into a disk
- Every cycle in Σ is homotopic to a concatenation of these loops. *[Jordan 1866]*
- Dehn *assumes* that the input cycle is a walk in this graph.



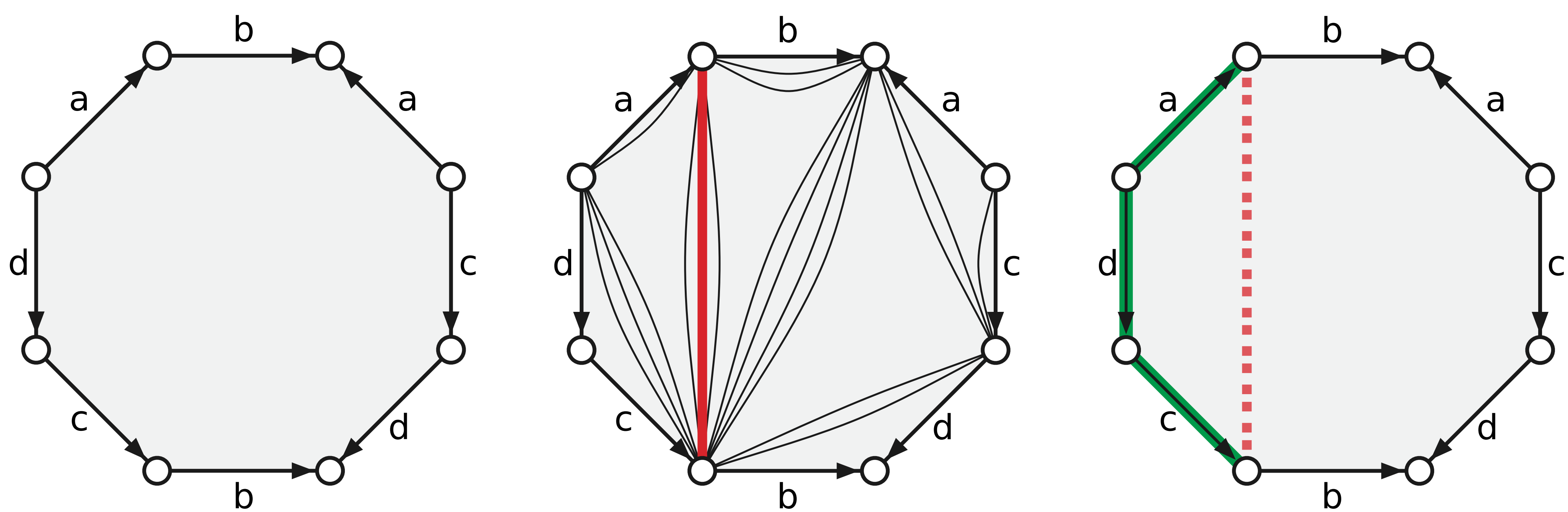
System of loops

- Contract edges between distinct vertices (spanning tree)
- Delete edges separating distinct faces (spanning cotree)
- Euler's formula $\Rightarrow$ exactly $2g$ edges left



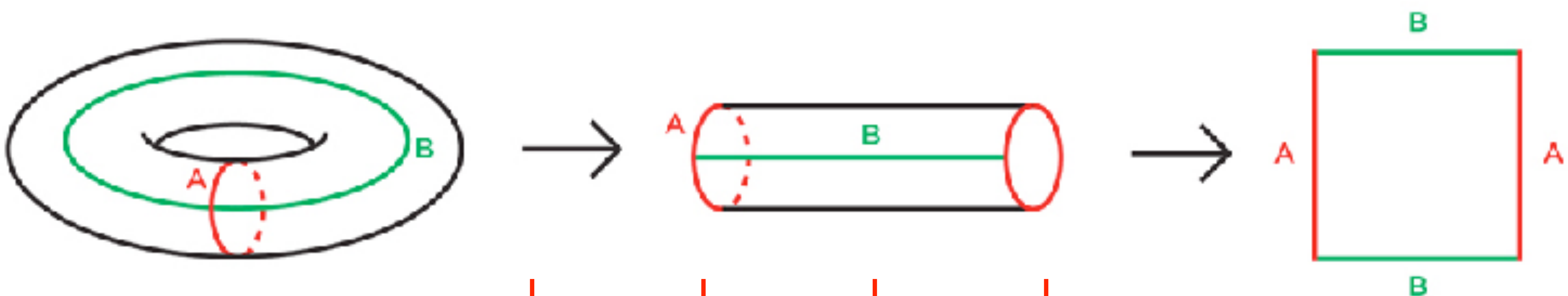
System of loops

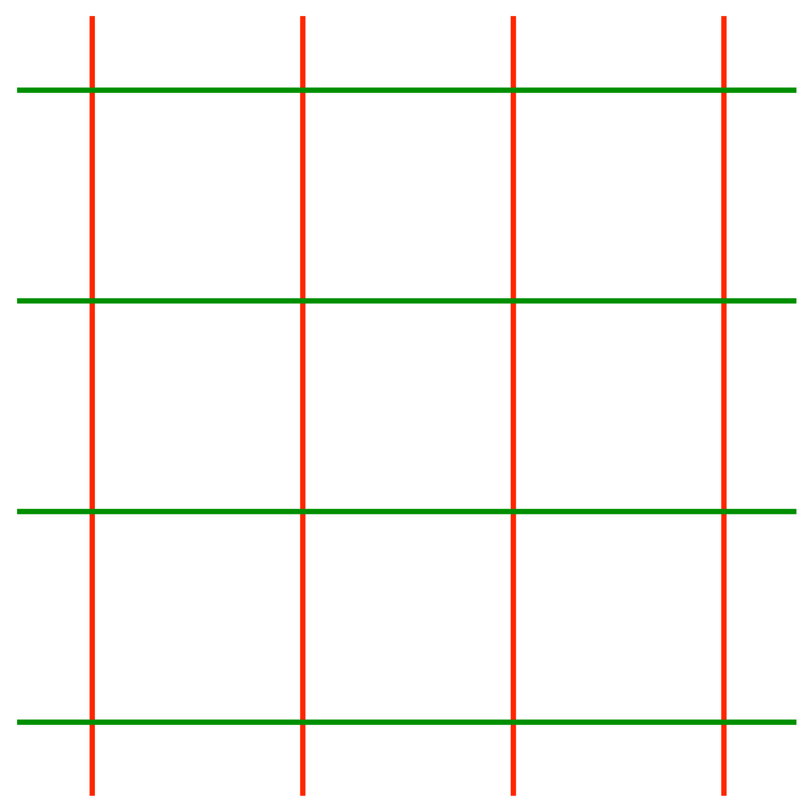
- Cut the surface along 2g loops $\longrightarrow$ *fundamental polygon P* with $4g$ edges
- Edge in G → path of length $\leq 2g$ around boundary of P
- Walk of length ℓ in G → walk of length $\ell' \leq 2g\ell$ in loop system



Universal cover $\tilde{\Sigma}$

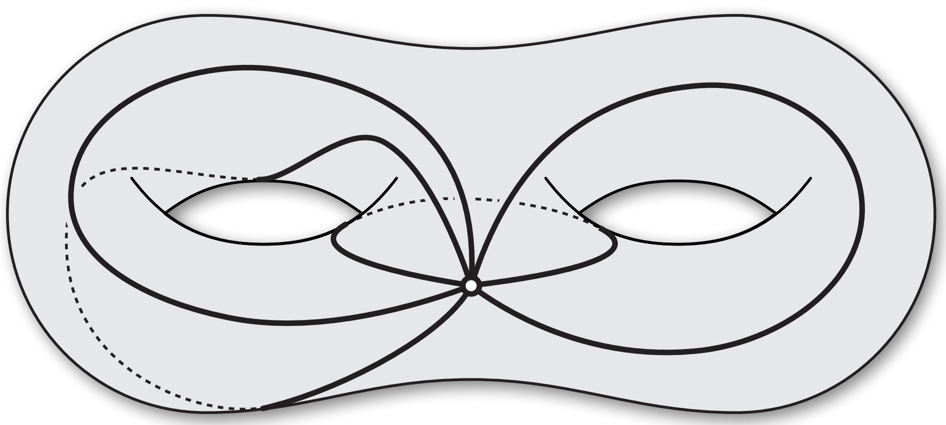
- Cut surface along loops $\longrightarrow$ *fundamental polygon*
- Glue infinitely many fundamental polygons into a plane.

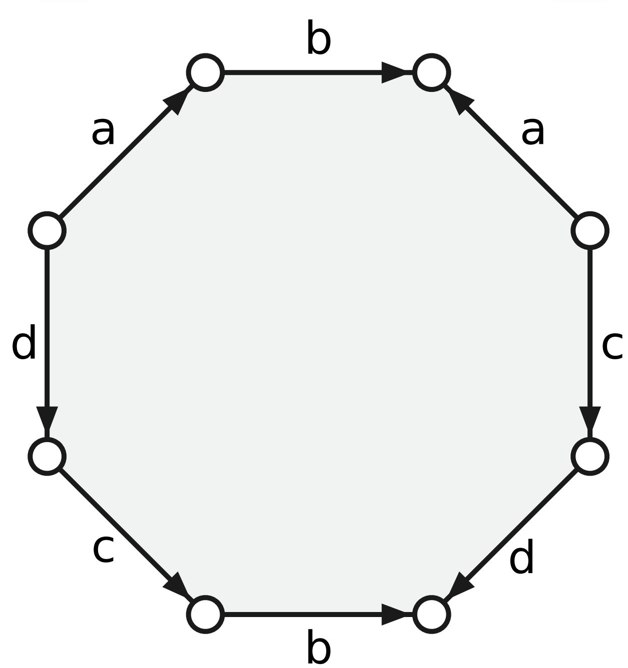


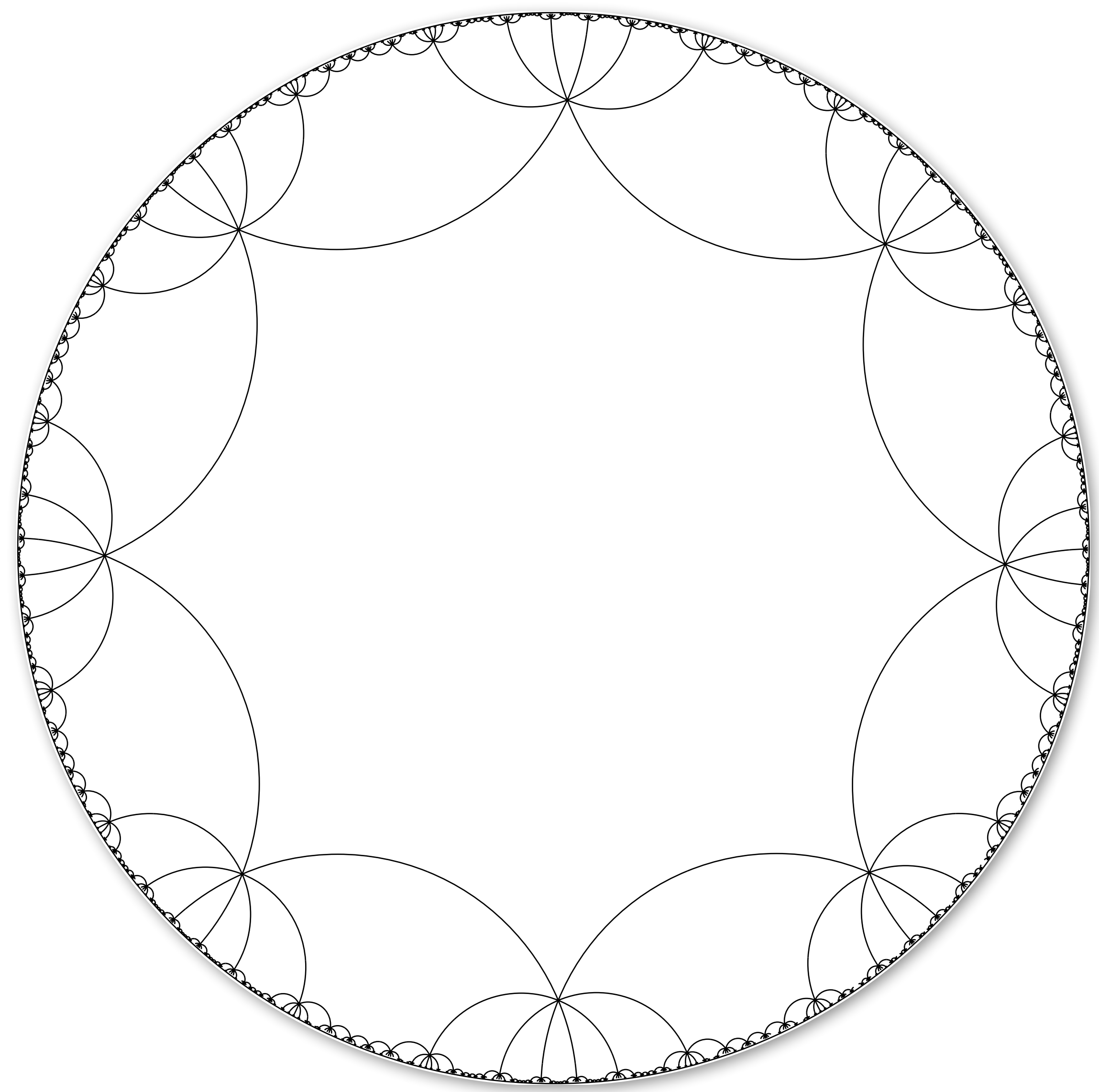


Universal cover $\tilde{\Sigma}$

When $g>1$, the universal cover is a regular tiling of the *hyperbolic* plane by $4g$-gons with vertices of degree $4g$.







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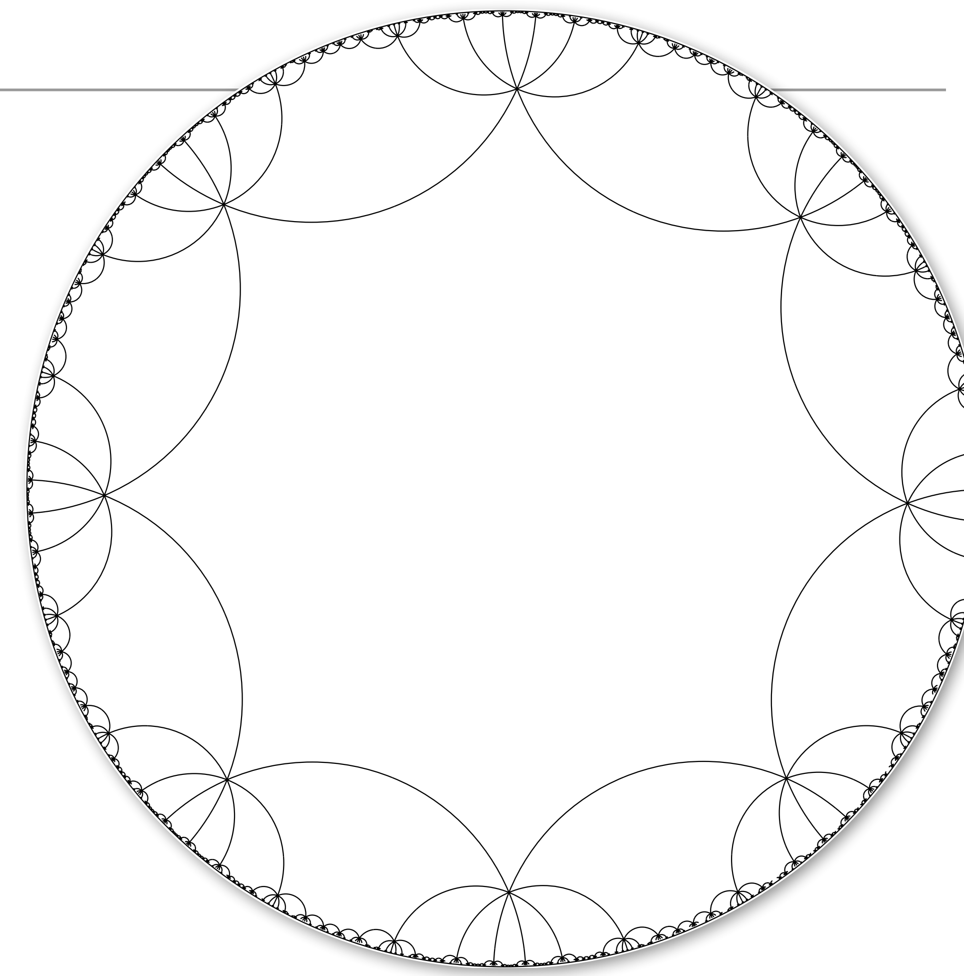
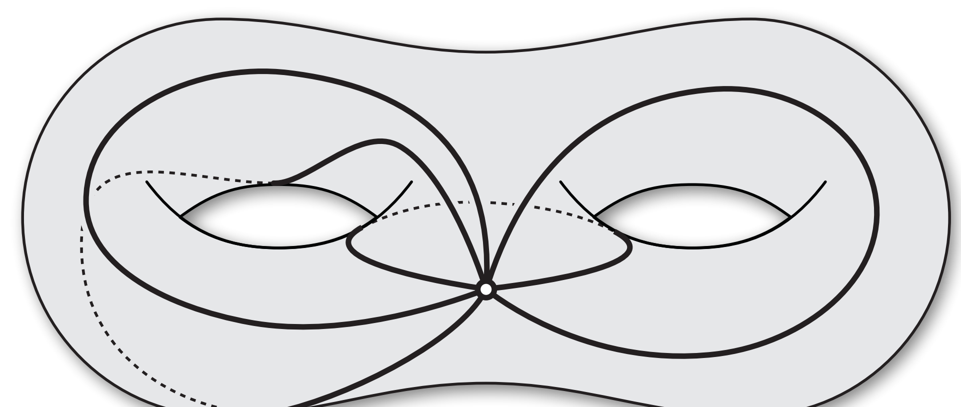


- Any point in Σ has infinitely many *lifts* in the universal cover, one for each homotopy class of loops.
- Every path has infinitely many lifts, each determined by a lift of one endpoint. Just follow the "same" sequence of edges.
- A loop in Σ is contractible iff it lifts to a *loop* in $\tilde{\Sigma}$.



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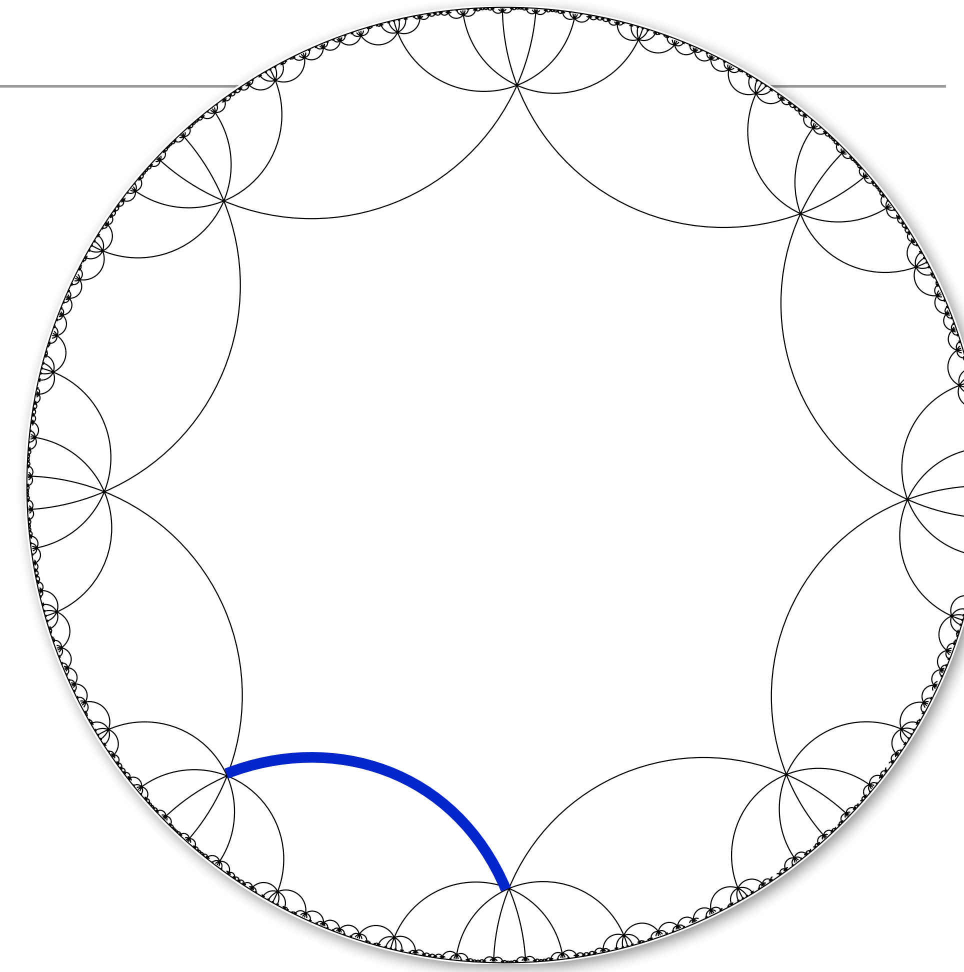

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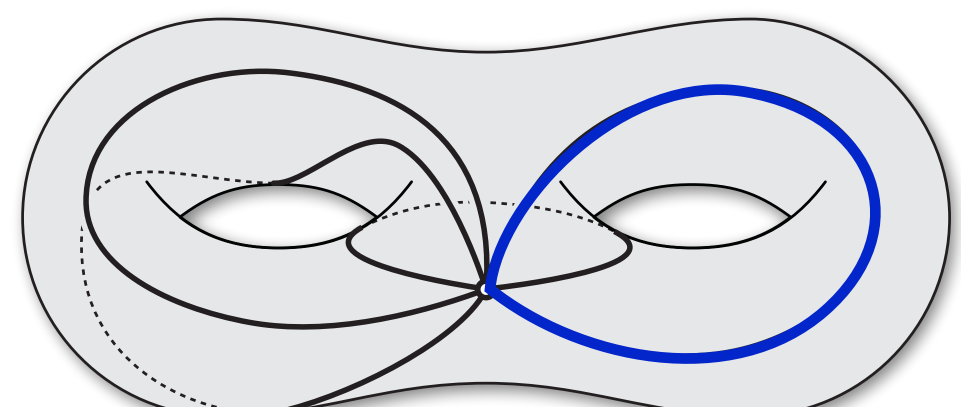

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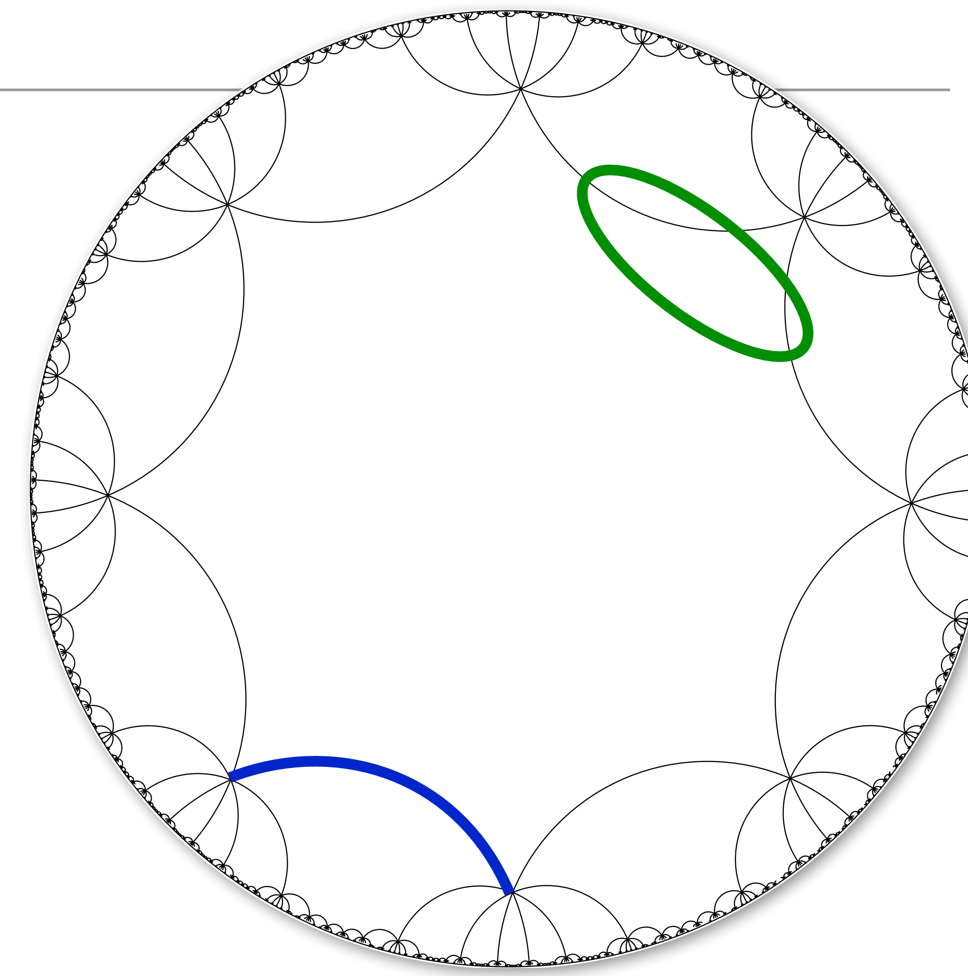

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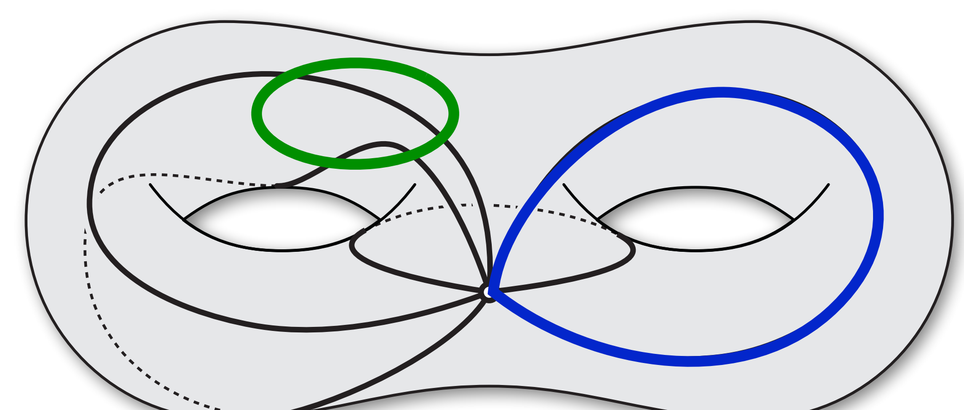
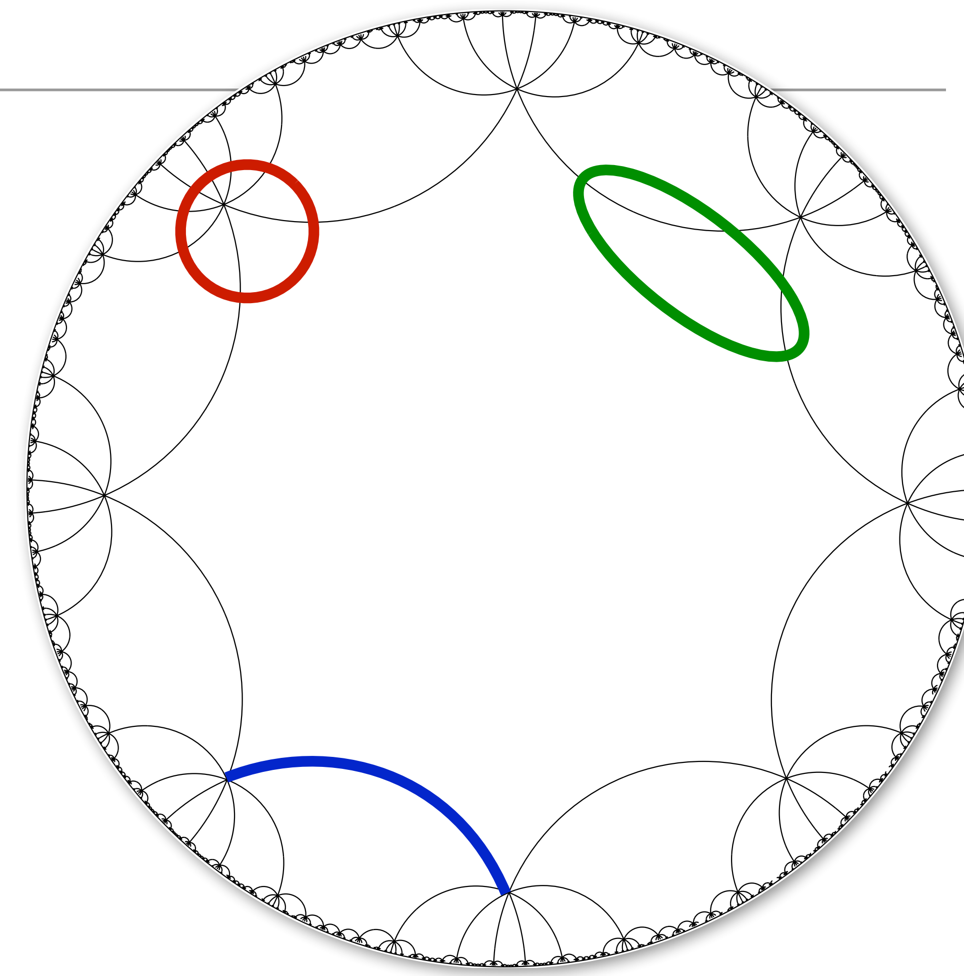
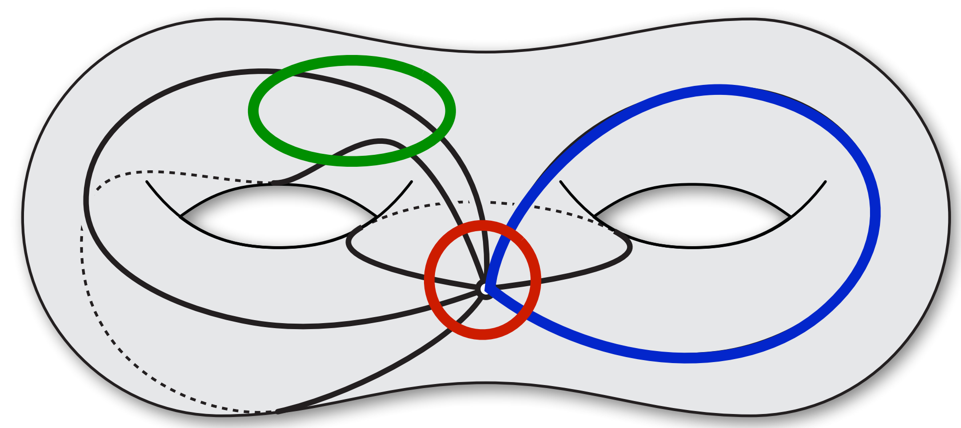


Universal cover $\tilde{\Sigma}$



- Any point in Σ has infinitely many *lifts* in the universal cover, one for each homotopy class of loops.
- Every path has infinitely many lifts, each determined by a lift of one endpoint. Just follow the “same” sequence of edges.
- A loop in Σ is contractible iff it lifts to a *loop* in $\tilde{\Sigma}$.



Dehn's algorithm

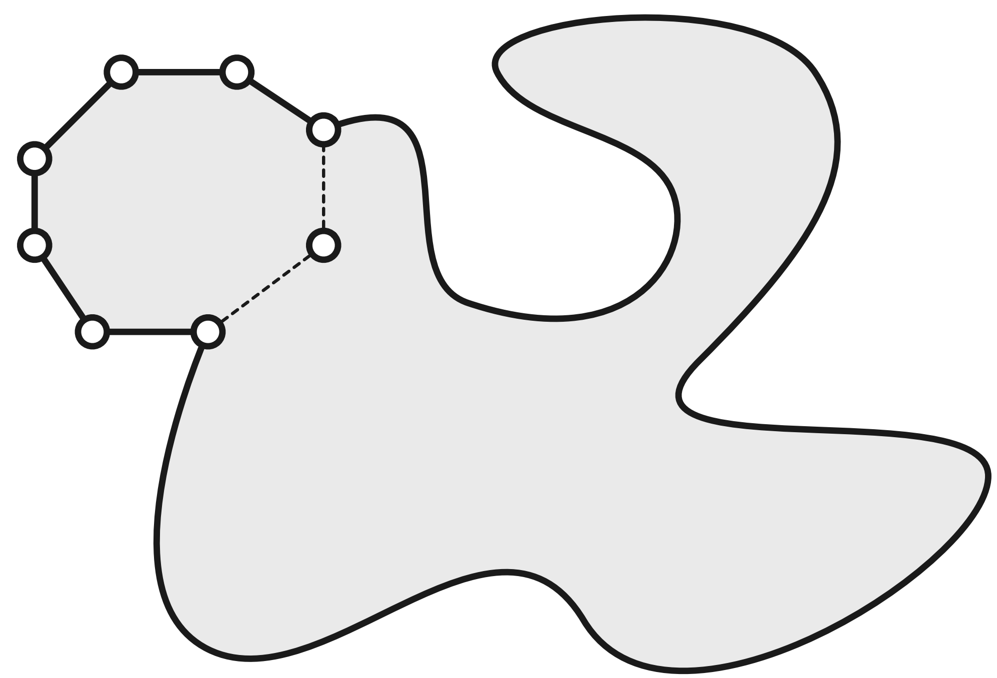
[Dehn 1912]

- **Goal:** Find a *shortest* cycle homotopic to the given cycle a.
- **Algorithm:** Greedily *reduce* a by local curve shortening.
- Hyperbolic structure ⇒ *no local minima!*
- Thus a is contractible iff it reduces to nothing.

Dehn's algorithm

[Dehn 1912]

- **Key Lemma:** Every nontrivial closed walk in $\tilde{\Sigma}$ contains either a *spur* or $4g-2$ consecutive edges of some face.



- Thus, we can locally shorten any contractible cycle.

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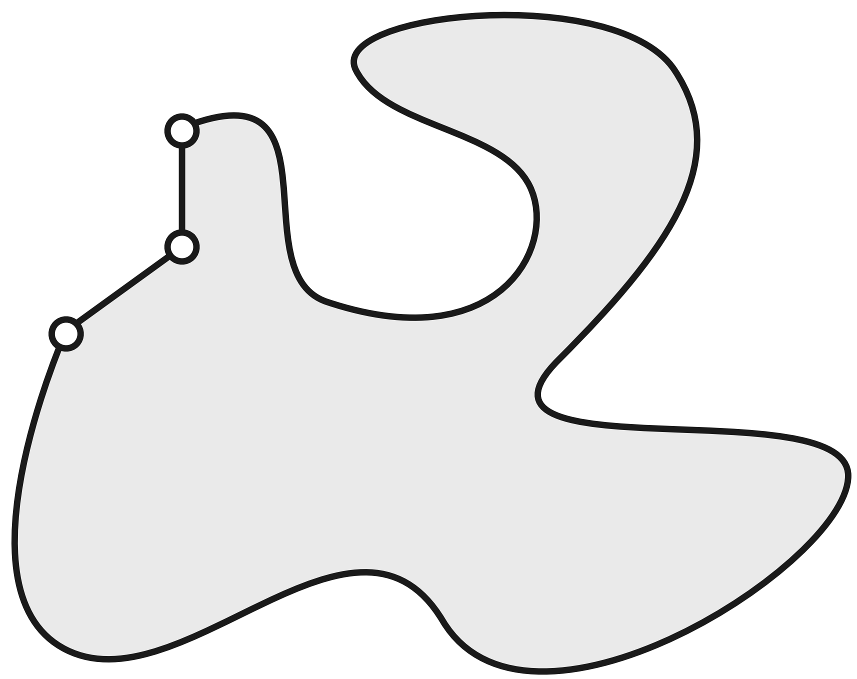


- Thus, we can locally shorten any contractible cycle.

René Descartes (c.1630)

Progymnasmata De Solidorum Elementis

Ponam ſemper pro numero angulorum ſolidorum α & pro numero facierum φ. Aggregatum ex omnibus angulis planis eſt $4\,\alpha - 8$, & numerus φ eſt $2\,\alpha - 4$, ſi numerentur tot facies quot poſſunt eſſe triangula. Numerus item angulorum planorum eſt $6\,\alpha - 12$, numerando ſcilicet vnum angulum pro tertiâ parte duorum rectorum. Nunc ſi ponam $3\,\alpha$ pro tribus angulis planis qui ad minimum requiruntur vt componant vnum angulum angulorum ſolidorum, ſuperſunt $3\,\alpha - 12$, quæ ſumma addi debet ſingulis angulis ſolidis juxta tenorem quæſtionis, ita vt æqualiter omni ex parte diffundantur. Numerus verorum angulorum planorum eſt $2\,\varphi + 2\,\alpha - 4$, qui non debet eſſe major quàm $6\,\alpha - 12$; ſed ſi minor eſt, exceſſus erit $+ 4\,\alpha - 8 - 2\,\varphi$.

René Descartes (c.1630)

Progymnasmata De Solidorum Elementis

Let α always denote the number of solid angles and φ the number of faces. **The total of all plane angles is 4α−8 [right angles], and the number φ is 2α−4, if as many faces as possible are triangles.** The number of planar angles themselves is 6α−12, counting for each angle a third part of two right angles. Then if I take 3α for the three planar angles that are required at minimum to comprise one angle of a solid angle, there remain 3α−12 that must be added to the solid angles, according to the terms of the question, so that they are distributed equally to all parts. **The total number of plane angles is 2φ−2α−4**, which cannot be larger than 6α−12; if it is less, the excess is 4α−8−2φ.

Combinatorial Gauss-Bonnet Theorem

- Consider any surface map (V, E, F), possibly with boundary
- Assign an arbitrary *exterior angle* $\angle c$ to every corner c
- Vertex *curvature*: $\kappa(v) := 1 - \frac{1}{2}\deg(v) + \sum_{c \in v} \angle c$
- Face *curvature*: $\kappa(f) := 1 - \sum_{c \in f} \angle c$

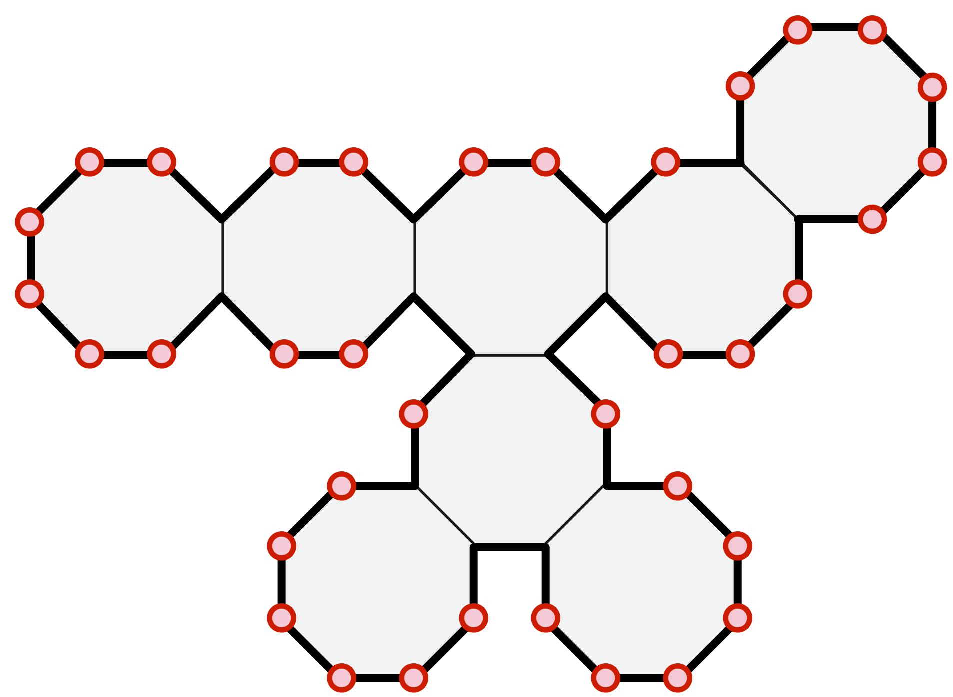
$$\sum_v \kappa(v) + \sum_f \kappa(f) = \chi = V - E + F$$

[Descartes 1630, Hilbert Cohn-Vossen 1932, Pólya 1954, Lyndon 1966, Banchoff 1967, Gromov 1987, McCammond Wise 2000, ...]

Dehn's lemma

[Dehn 1912] [Lyndon 1966]

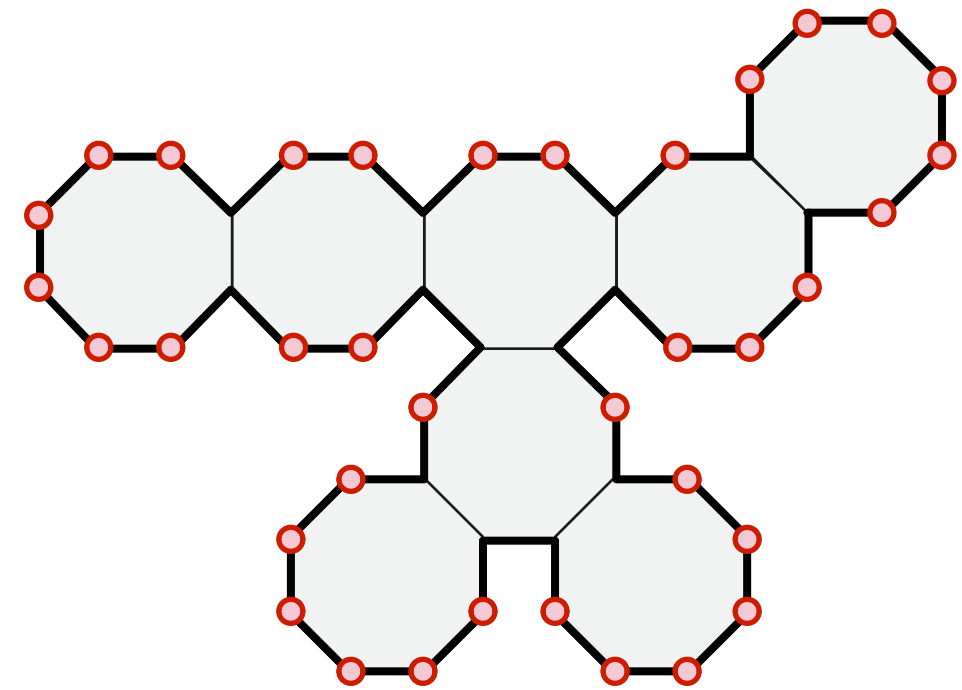
- Consider a *simple* cycle in $\tilde{\tilde{\Sigma}}$ — no spurs, no self-intersections. (This is the interesting case.)
- Call a vertex *convex* if it is incident to exactly one interior corner. Let V^+ be the set of convex vertices.



Dehn's lemma

[Dehn 1912] [Lyndon 1966]

- Assign angle ¼ to every corner.
  - $\kappa(v) = ¼$ for every convex vertex v
  - $\kappa(v) \leq 0$ for every non-convex vertex v
  - $\kappa(f) = 1-g < 0$ for every face f



- Discrete Gauss-Bonnet ⇒ $\sum_v \kappa(v) + \sum_f \kappa(f) = 1$

  ⇒ $|V^+| \geq (4g-4)|F| + 4$.

  ⇒ Some face has $\geq 4g-3$ consecutive convex corners. □

Analysis

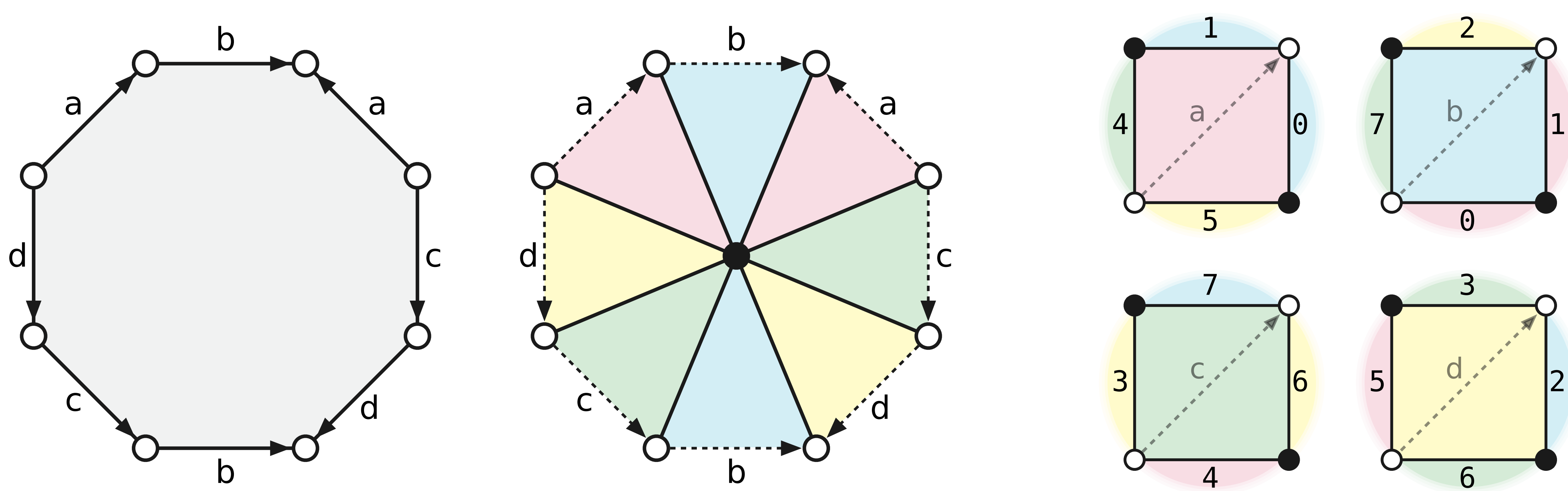
- Time for greedy reduction:
  - Brute force: $O(g^2)$ time per edge
    - At each step, compare last $2g-2$ edges to $O(g)$ patterns
  - Smarter: $O(1)$ amortized time per edge
    - Use a DFA for pattern matching
    - Charge $O(g)$ modification time to the $2g-4$ edges removed
- Total time for reduction is $O(\ell')$
- So overall time is $O(gn + \ell') = O(gn + g\ell)$

Linear time contractibility

System of quads

[Lazarus Rivaud 2012]

- Add edges from center of P to corners, delete edges of P
- 2 vertices, $4g$ edges, $2g$ quadrilateral faces
- Edge in Σ → path of length ≤ 2 inside P
- Walk of length ℓ in Σ → walk of length $\ell' \leq 2\ell$ in quad system



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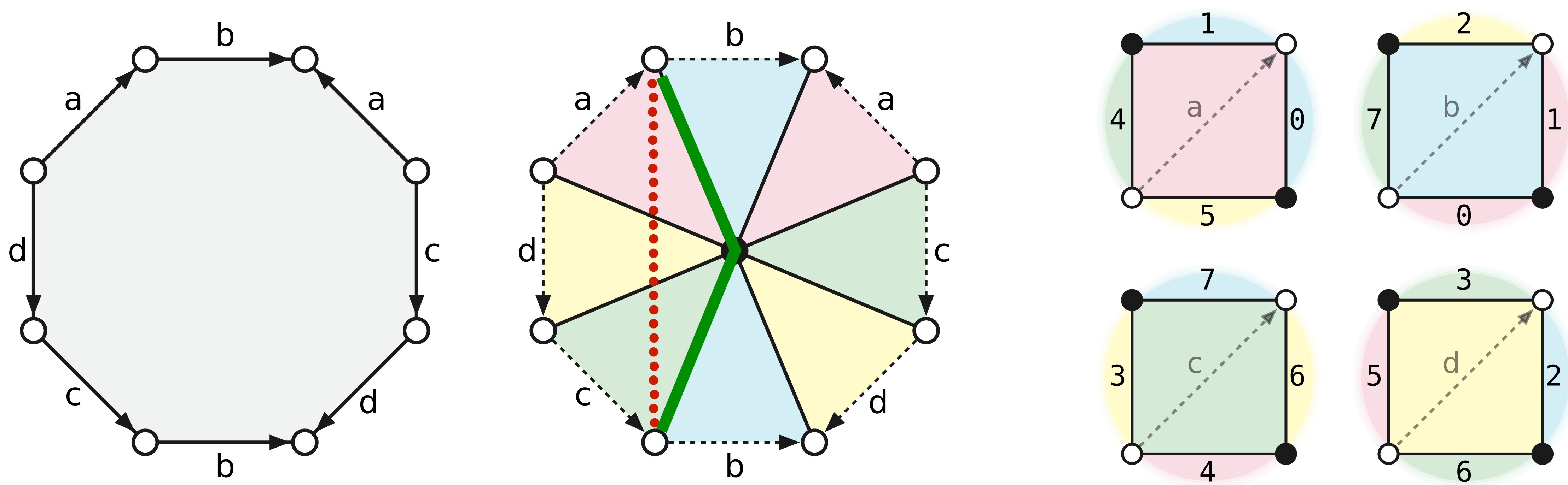
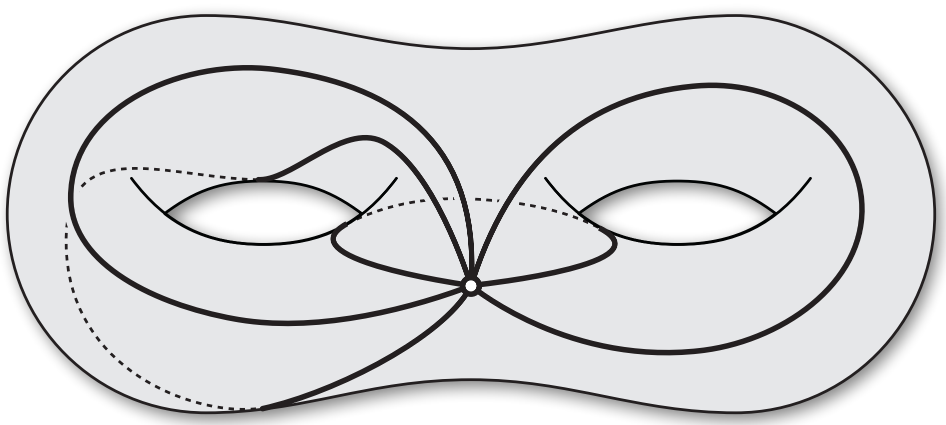
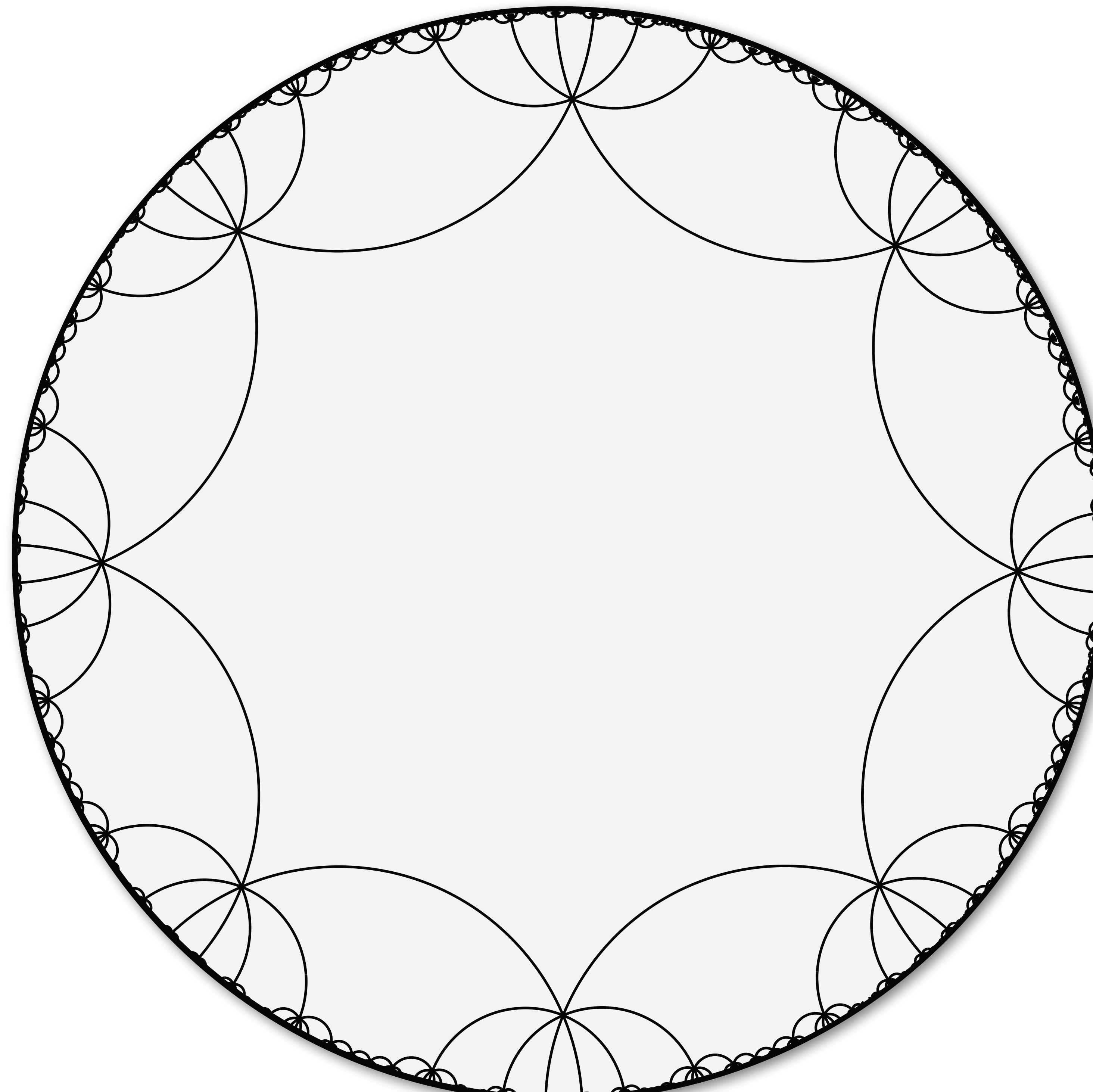


Regular hyperbolic structure





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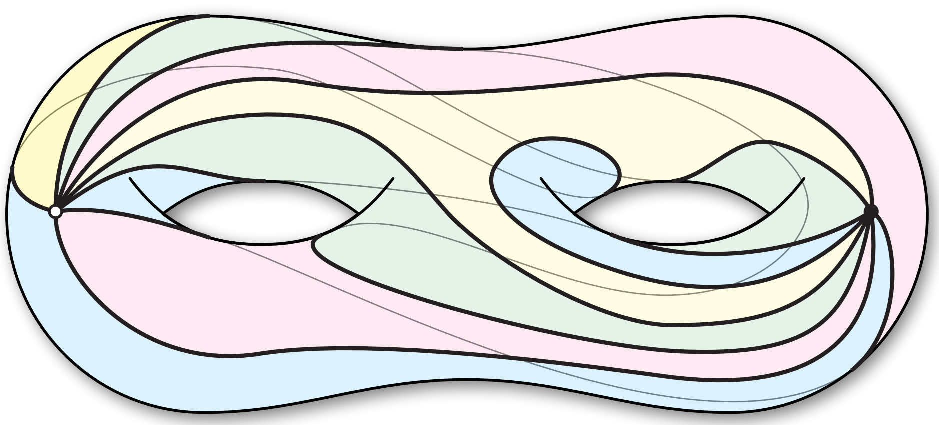
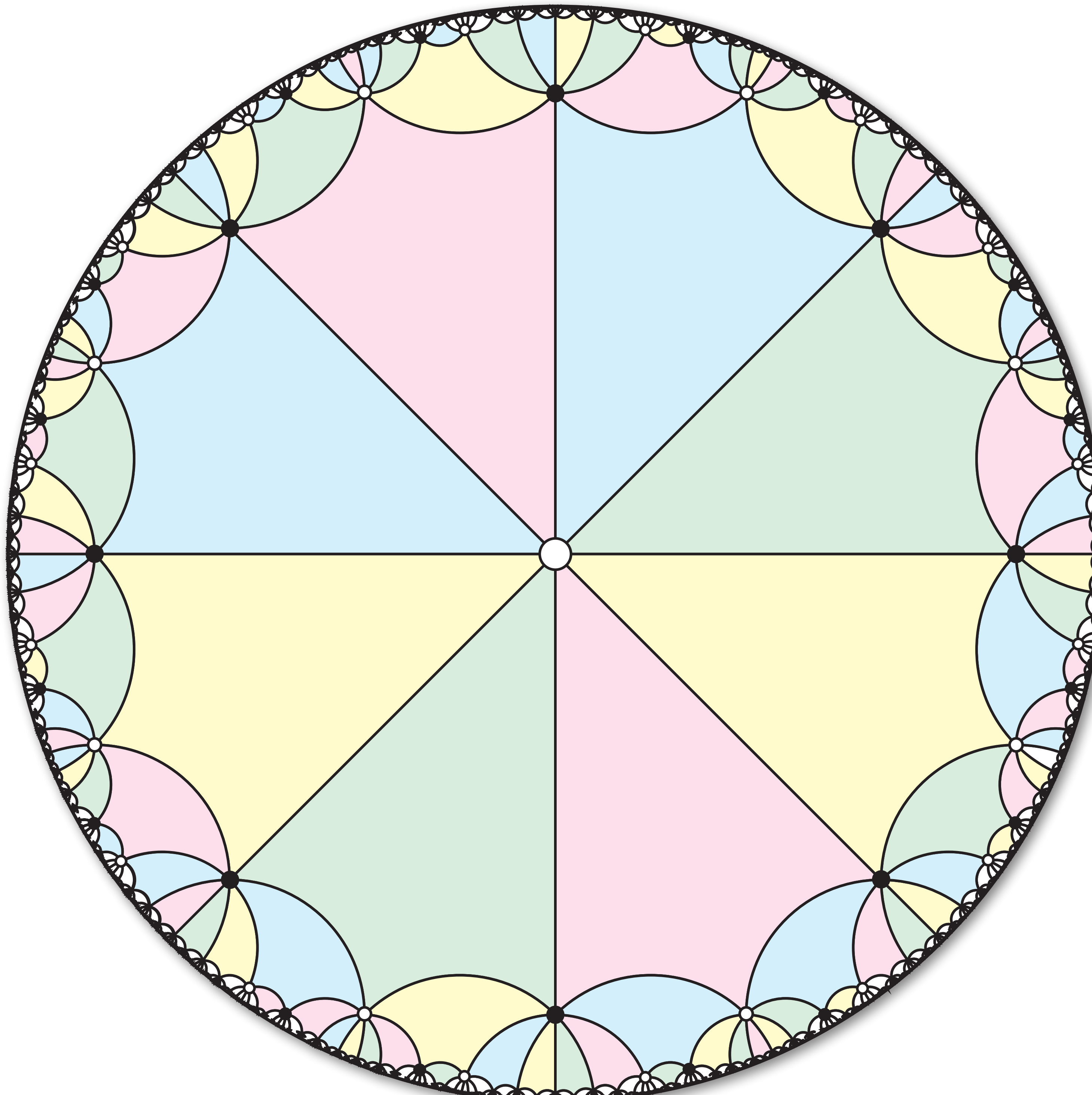


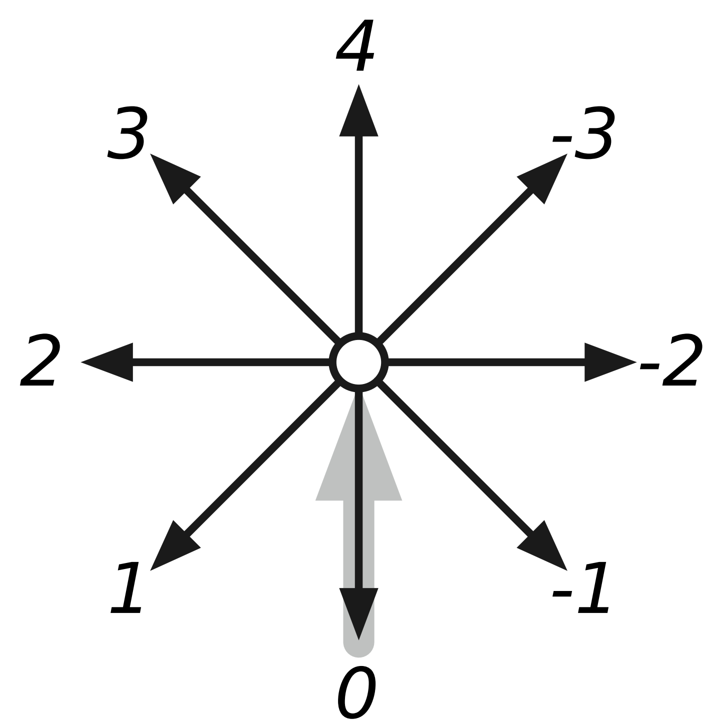


Dehn's algorithm

- **Goal:** Find a *shortest* cycle homotopic to the given cycle α.
- **Algorithm:** Greedily *reduce* α by local curve shortening.
- Hyperbolic structure $\Rightarrow$ *no local minima*!
- Thus α is contractible iff it reduces to nothing.

Turn sequence

- Consider a closed walk $(v_0, e_1, v_1, e_2, \ldots, v_{\ell-1}, e_\ell)$.
- *Turn* τ_i = number of corners between e_{i-1} and e_i in clockwise order around v_i.



- Turn sequence $(\tau_0, \tau_1, \ldots, \tau_\ell)$ is easy to compute in $O(\ell)$ time.

Run-length encoding

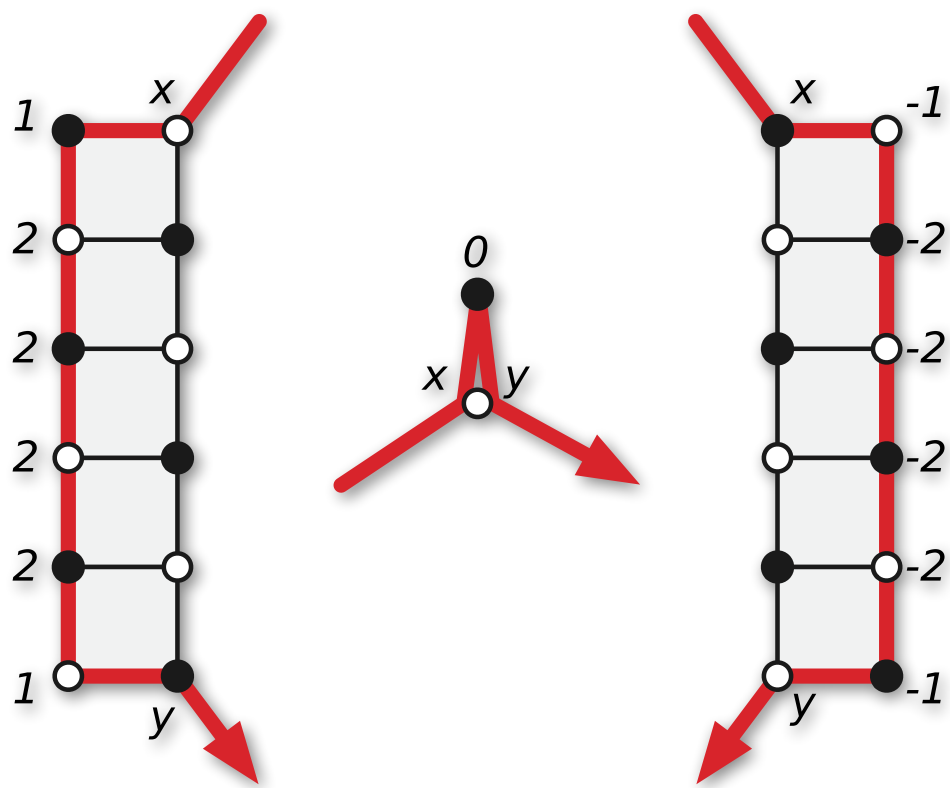
- Record lengths of maximal runs of equal turns

$$(1, \mathbf{2^4}, 1^2, 2) \;=\; (1, \mathbf{2, 2, 2, 2}, 1, 1, 2)$$

- Easy to compute in $O(\ell)$ time
- The rest of the algorithm manipulates *only* run-length encoded turn sequences.

Spurs and brackets

- Left bracket: $1\ 2^k\ 1$
- Spur: 0
- Right bracket: $-1\ -2^k\ -1$

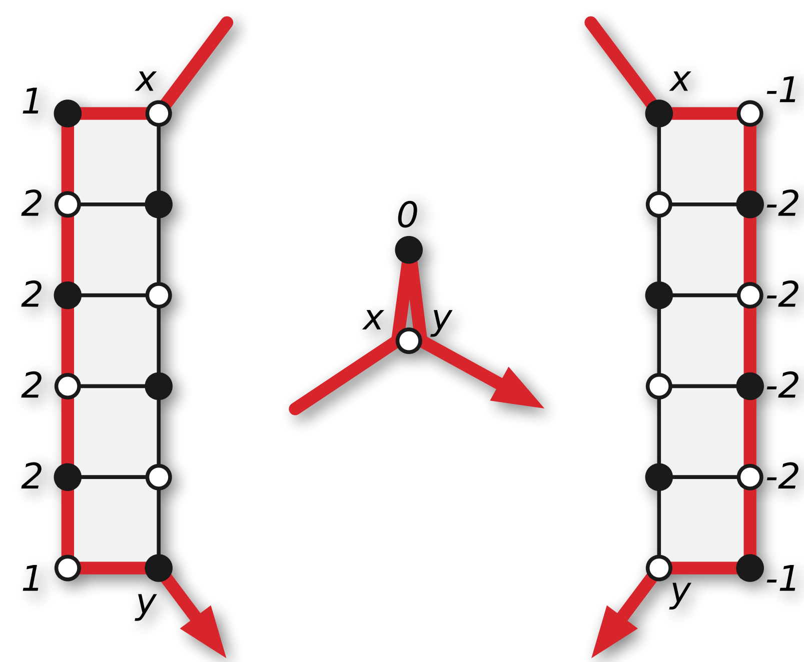


Bracket Lemma

[Gersten Short 1990]

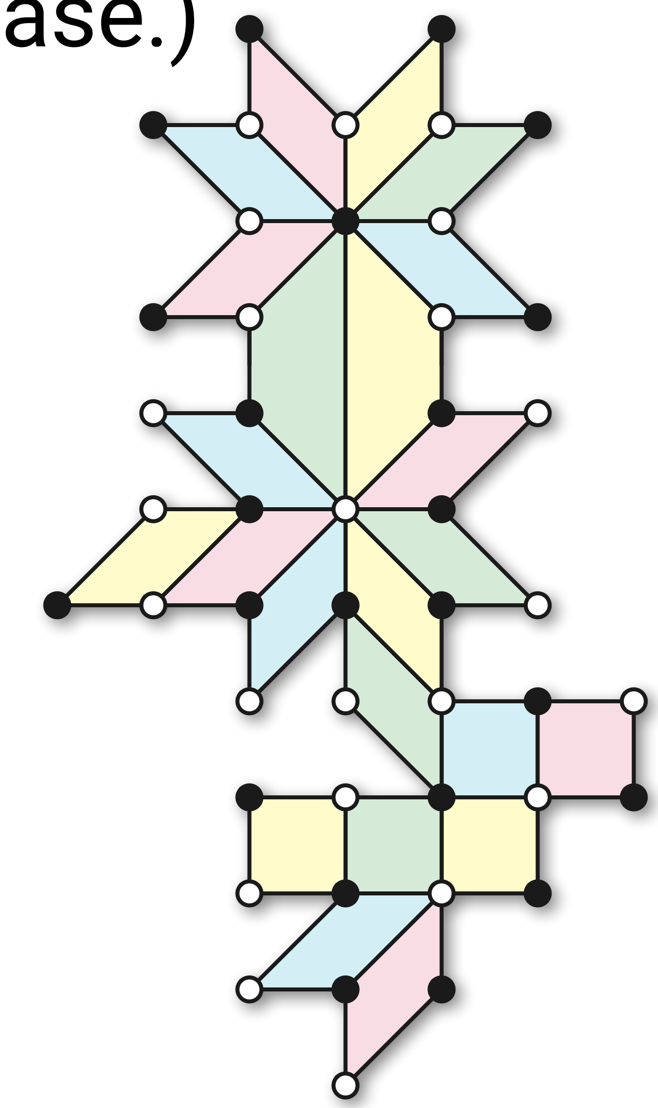
- *Every nontrivial contractible cycle in $\tilde{Q}$ contains either a spur or a bracket.*

- **Corollary:** *A cycle is contractible iff it reduces to nothing.*



Bracket Lemma

- Consider a *simple* cycle in $\tilde{Q}$ — no spurs, no self-intersections. (This is the interesting case.)



- *Label boundary vertices as follows:*
  - *Convex: next to 1 corner*
  - *Flat: next to 2 corners*
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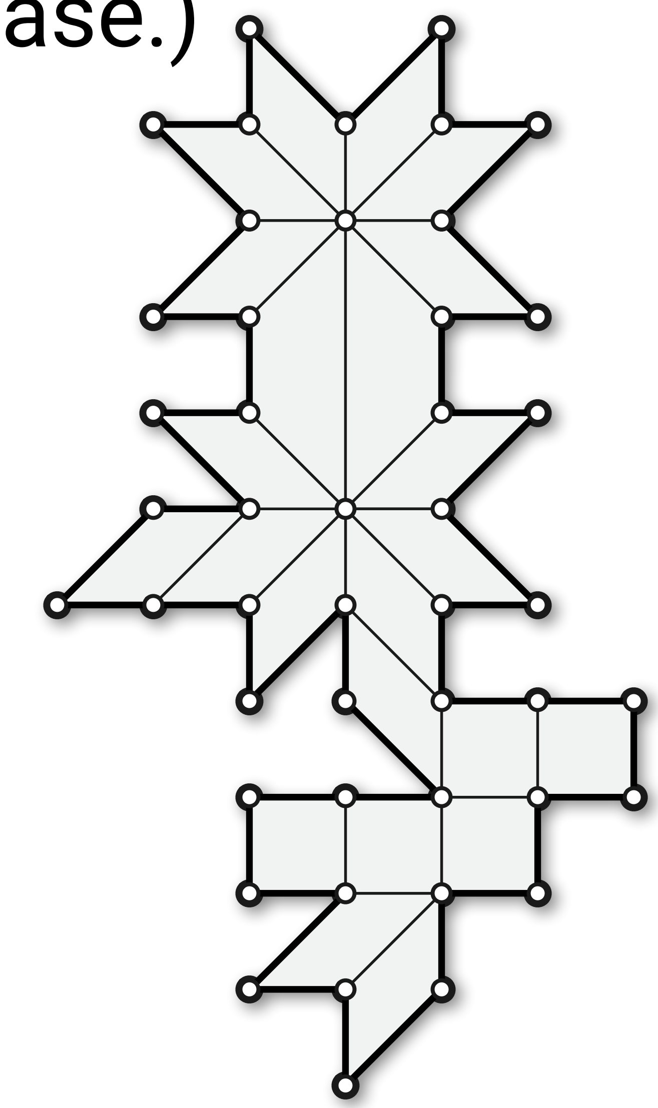

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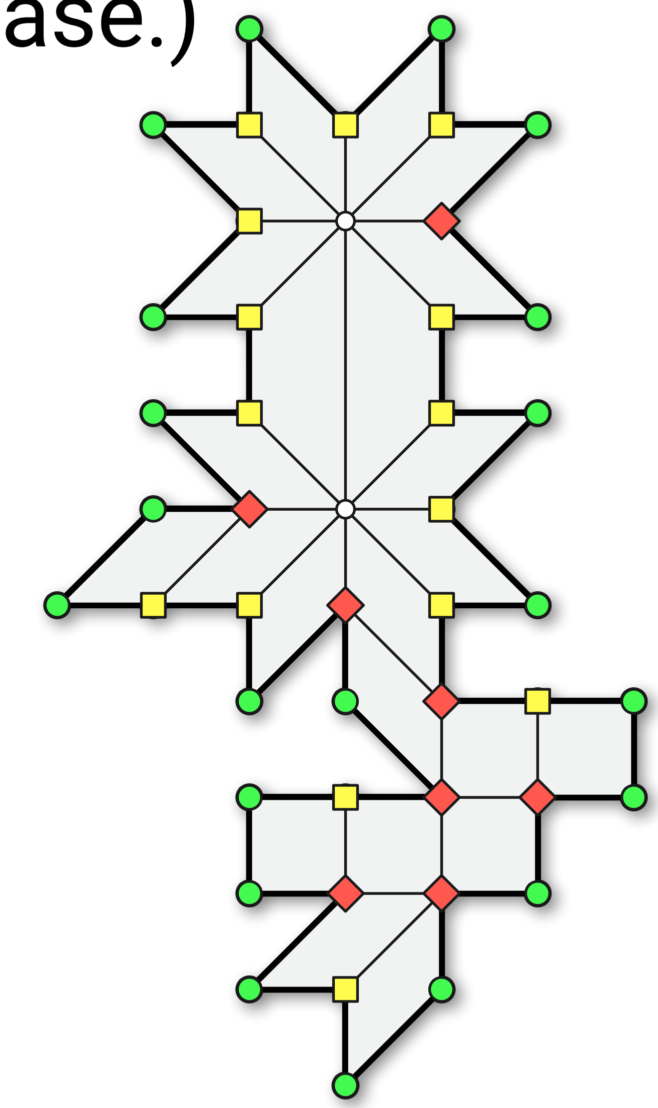

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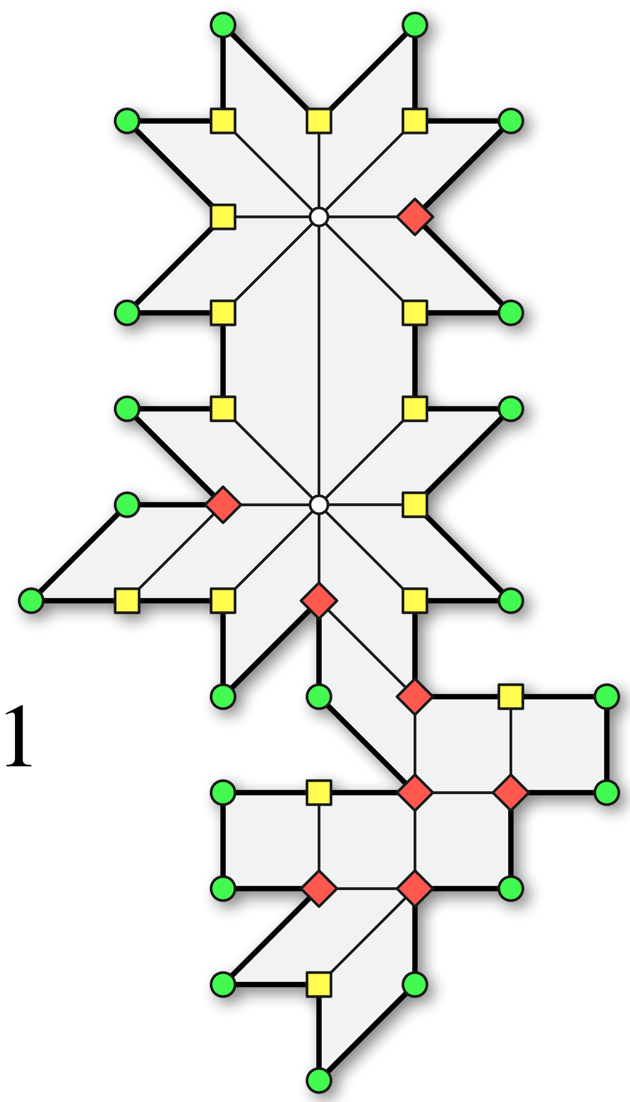
Bracket Lemma

- Assign angle ¼ to every corner. Faces are *squares*!
  - Every face has curvature 0
  - Convex vertices have curvature +¼
  - Flat vertices have curvature 0
  - Concave vertices have curvature ≤ −¼
  - Interior vertices have curvature ≤ −1



- Combinatorial Gauss-Bonnet: $\sum_v \kappa(v) = 1$
  - ⇒ #convex ≥ #concave + 4
  - ⇒ There are at least four brackets!

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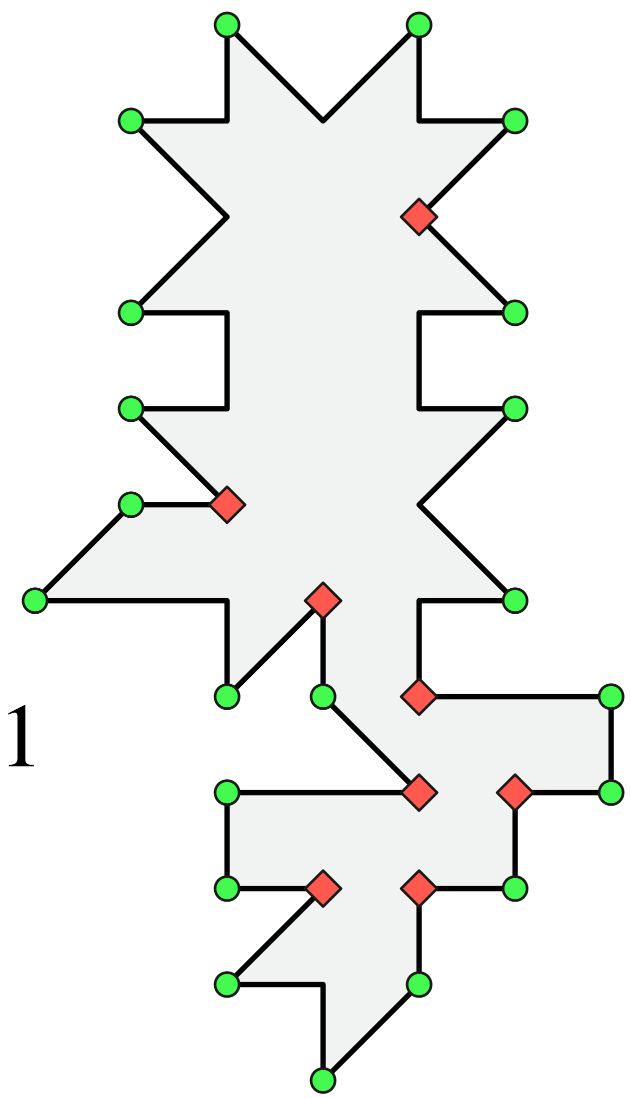

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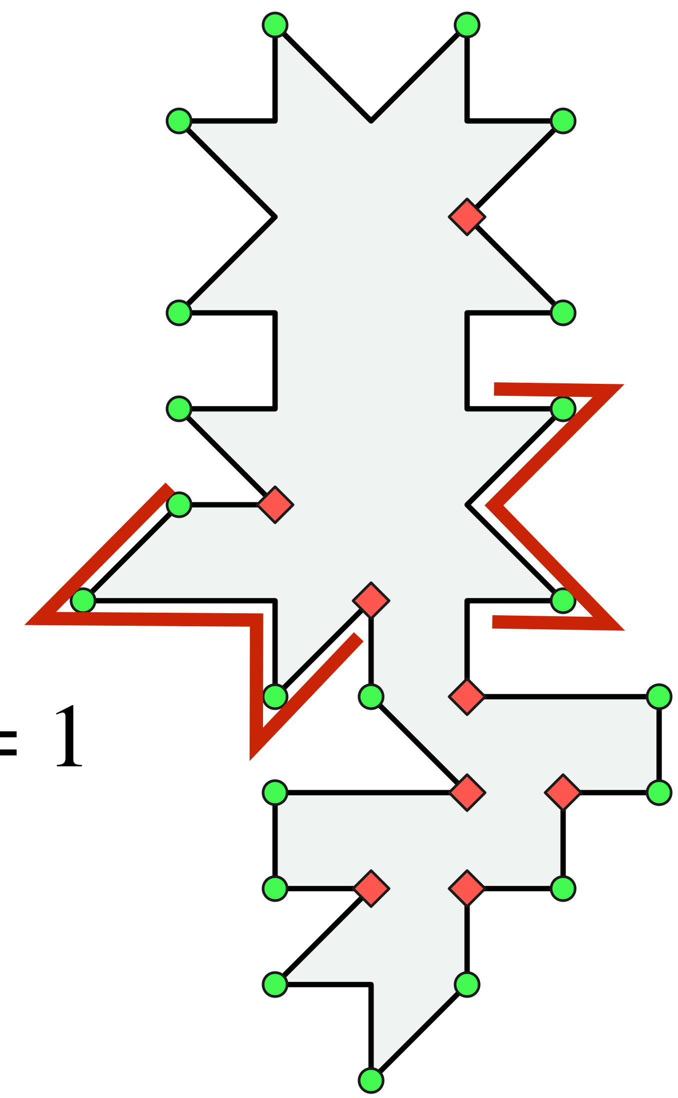

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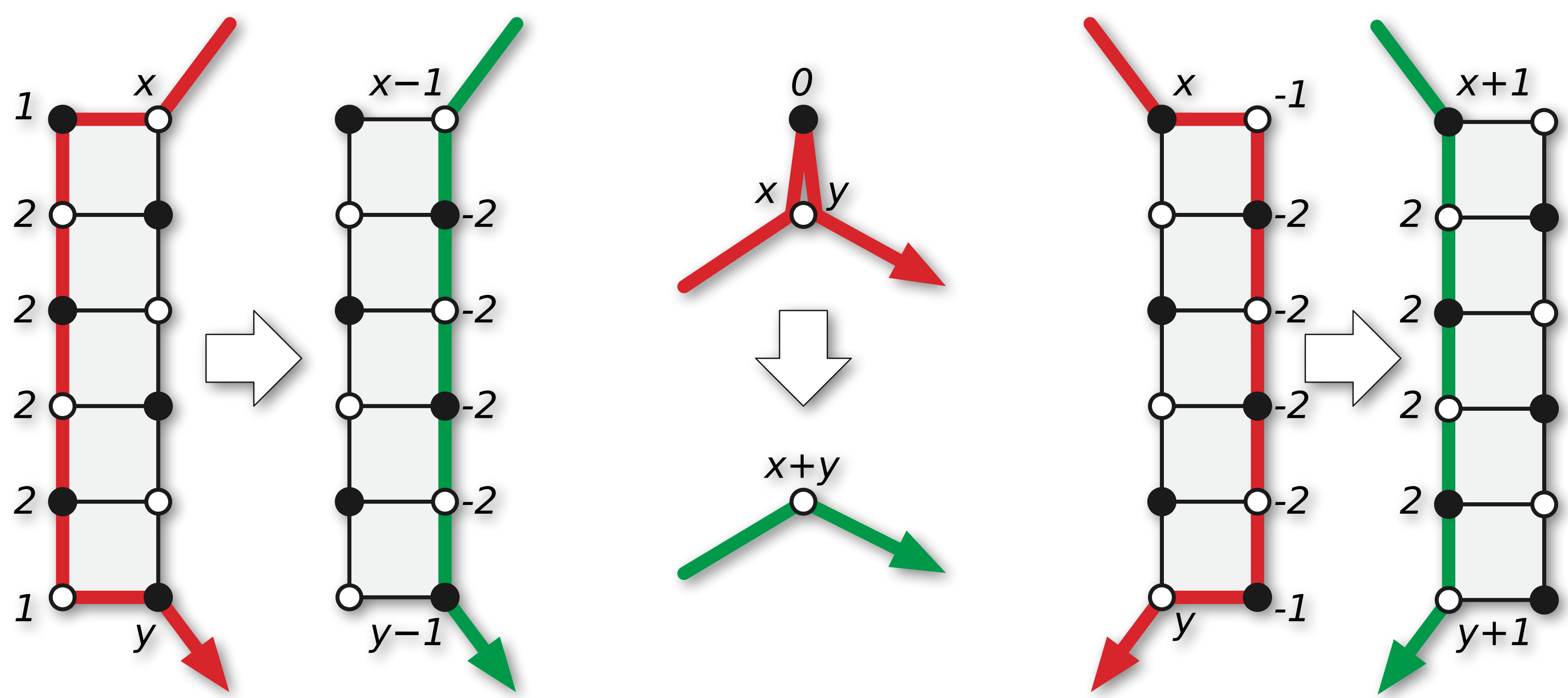
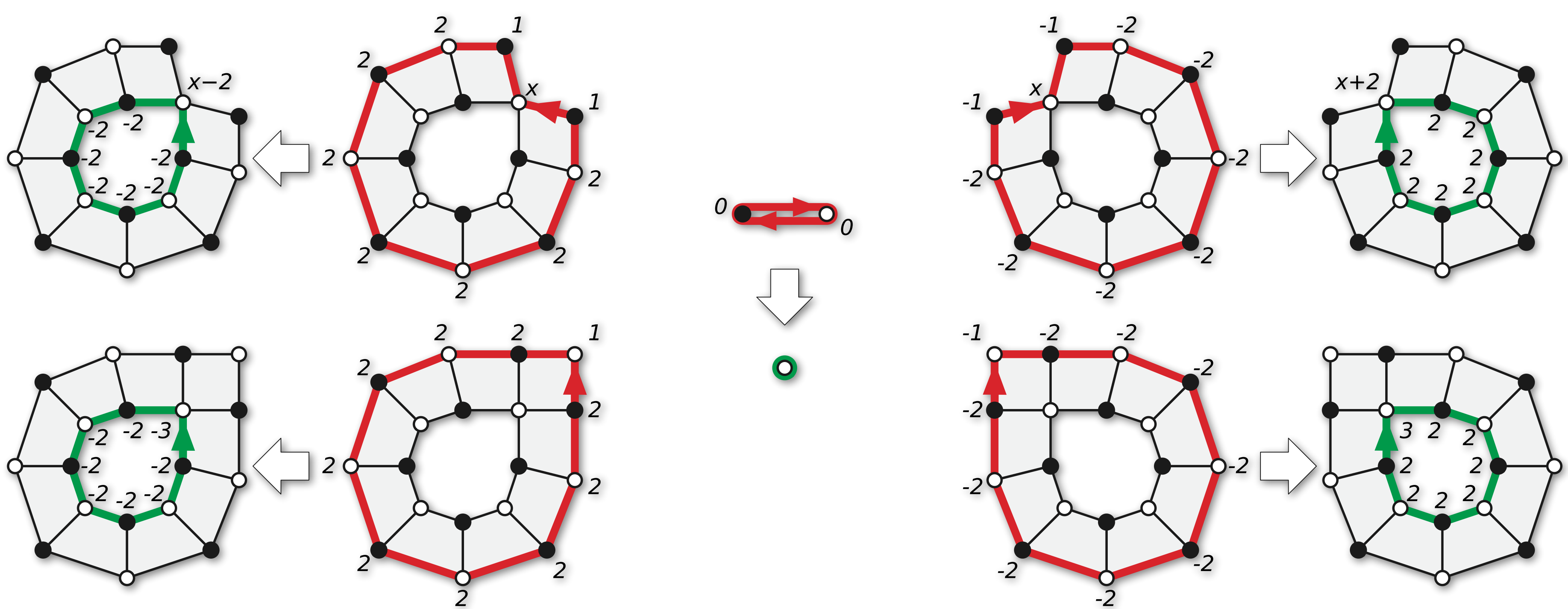
Elementary reductions

- Left bracket: $x\ \ 1\ \ 2^k\ \ 1\ \ y \rightarrow x{-}1\ \ {-2}^k\ \ y{-}1$
- Spur: $x\ \ 0\ \ y \rightarrow x{+}y$
- Right bracket: $x\ \ {-1}\ \ {-2}^k\ \ {-1}\ \ y \rightarrow x{+}1\ \ 2^k\ \ y{+}1$



Cyclic elementary reductions

- Left bracket: $(x\ \ 1\ \ 2^k\ \ 1) \rightarrow (x{-}2\ \ \text{-}2^k)$ or $(1\ \ 2^k) \rightarrow (\text{-}3\ \ \text{-}2^{k-2})$
- Double spur: $(0\ 0) \rightarrow (\ )$
- Right bracket: $(x\ \ \text{-}1\ \ \text{-}2^k\ \ \text{-}1) \rightarrow (x{+}1\ \ 2^k)$ or $(\text{-}1\ \ \text{-}2^k) \rightarrow (3\ \ 2^{k-2})$



Reduction algorithm

mark all runs *dirty*
$i \leftarrow 0$
repeat
    if runs $i-4$ .. i contain a spur or bracket
        perform an elementary reduction
        mark the modified runs *dirty*
        $i \leftarrow \max\{0, i-5\}$
    else
        mark run i *clean*
        $i \leftarrow i+1$
until all runs are marked *clean*

Reduction takes $O(\ell)$ time

- Turn sequence is run-length encoded

  $\Rightarrow$ Each iteration takes $O(1)$ time

- Each elementary reduction decreases length by 2

  $\Rightarrow$ At most $\ell/2$ elementary reductions

- Each elementary reduction creates or modifies ≤ 5 runs

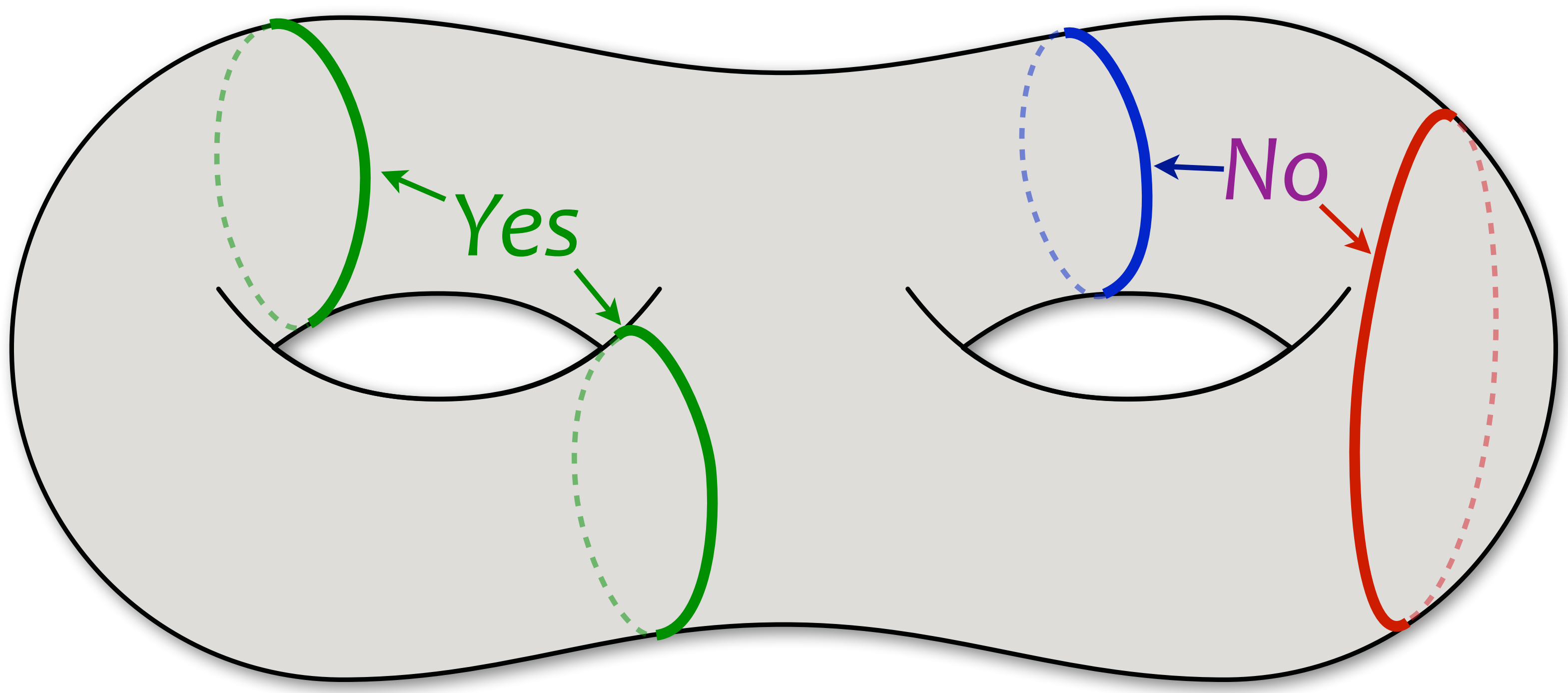
  $\Rightarrow$ At most $\ell + 5\ell/2$ runs to process

  $\Rightarrow$ At most 4ℓ iterations

Free homotopy

The homotopy problem

- Given two paths or cycles α and β in an orientable surface, can α be continuously deformed into β?



- Are α and β *freely homotopic* / in the same *free homotopy* class?

Free homotopy

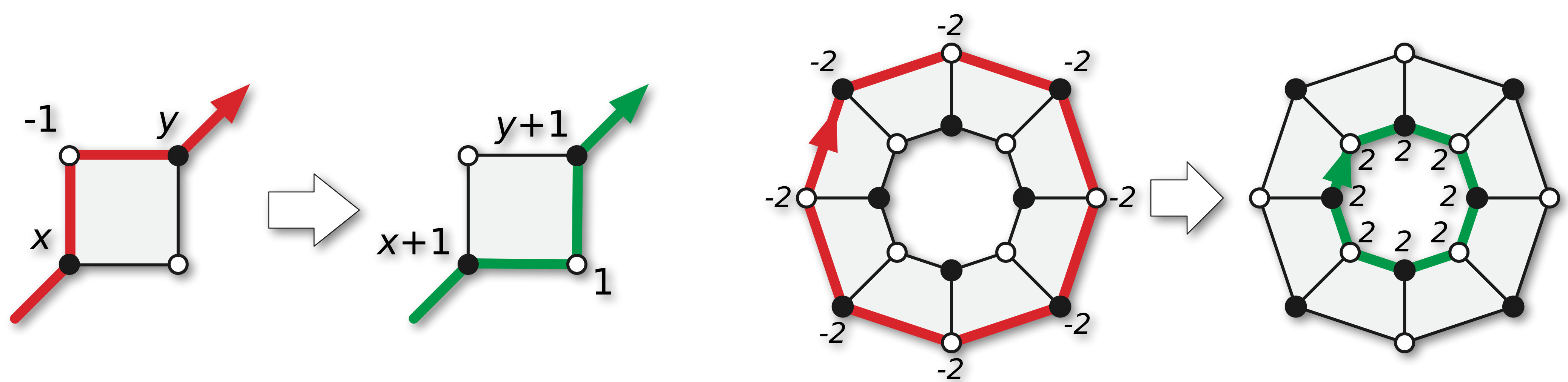
- **Goal:** Transform each cycle into the unique *canonical* cycle in its homotopy class. Two cycles are homotopic if and only if they yield the same canonical cycle.
  - With a *smooth* hyperbolic metric, each homotopy class has a unique shortest cycle. *[Dehn 1911-12]*
  - But in our *discrete* hyperbolic metric, shortest cycles are *not* unique.
- **Algorithm:** After shrinking the cycles, *shift them to the right* as far as possible without increasing their length.

[Lazarus Rivaud 2012]

Canonical cycles

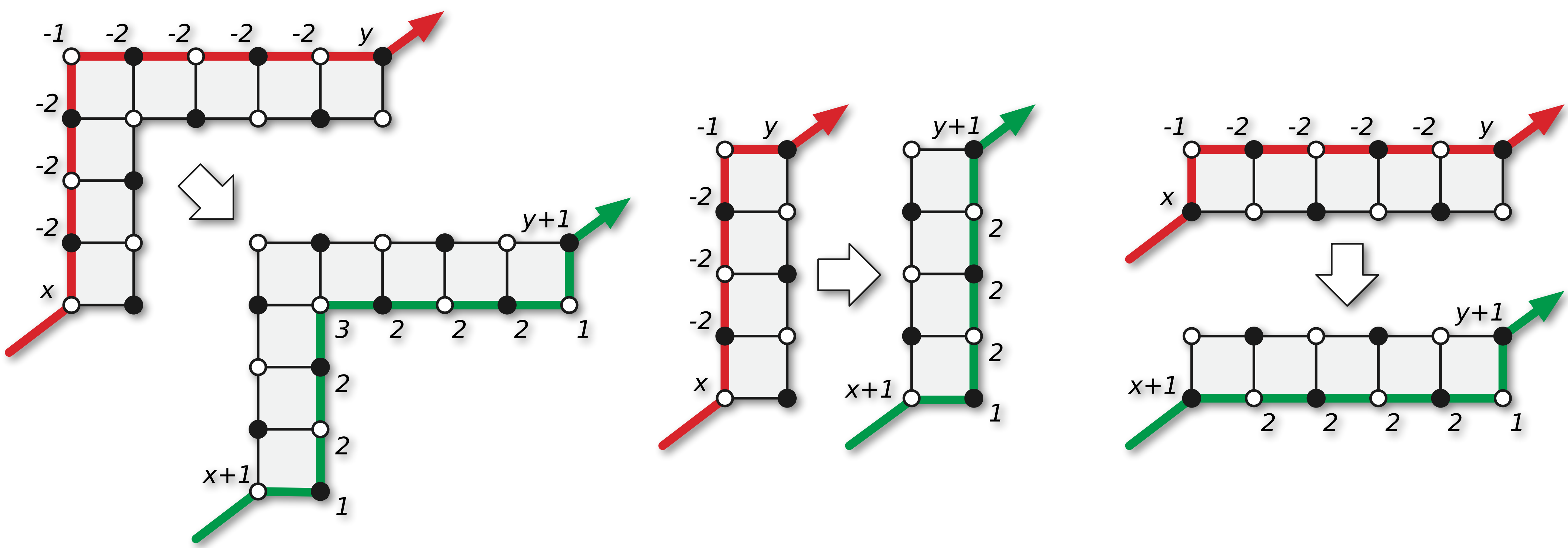
[Lazarus Rivaud 2012]
[Erickson Whittlesey 2013]

- A reduced cycle is *canonical* iff
  1. (1) it has no turn −1
  2. (2) not all turns are −2
- Canonical cycles are as far to the right as possible.



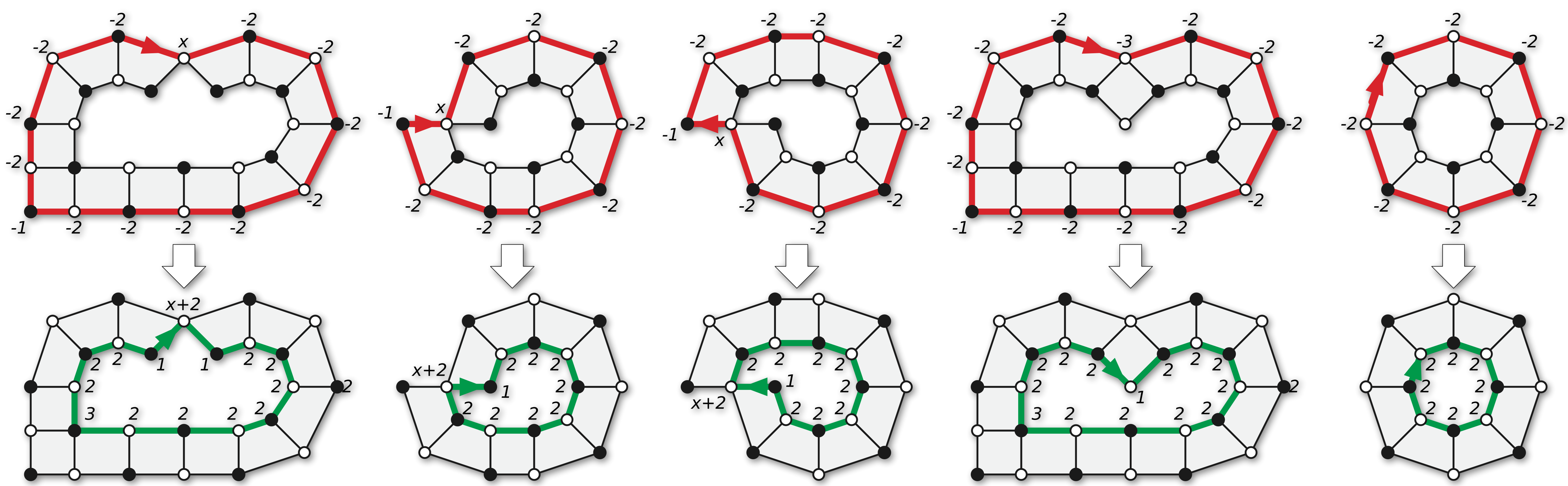
Elementary right shifts

- $x\ \ {-2}^s\ \ \mathbf{-1}\ \ {-2}^t\ \ y \;\rightarrow\; x{+}1\ \ 1\ \ 2^{s-1}\ \ 3\ \ 2^{t-1}\ \ 1\ \ y{+}1$ (if $s>0$, $t>0$)
- $x\ \ \mathbf{-1}\ \ {-2}^t\ \ y \;\rightarrow\; x{+}1\ \ 2^t\ \ 1\ \ y{+}1$
- $x\ \ {-2}^s\ \ \mathbf{-1}\ \ y \;\rightarrow\; x{+}1\ \ 1\ \ 2^s\ \ y{+}1$



Cyclic elementary right shifts

- $(x\ \ {-2}^s\ \ \mathbf{-1}\ \ {-2}^t) \rightarrow (x{+}2\ \ 1\ \ 2^{s-1}\ \ 3\ \ 2^{t-1}\ \ 1)$ *$(s>0,\ t>0,\ x\neq-3)$*
- $(x\ \ \mathbf{-1}\ \ {-2}^t) \rightarrow (x{+}2\ \ 2^t\ \ 1)$ *$(x\neq-3)$*
- $(x\ \ {-2}^s\ \ \mathbf{-1}) \rightarrow (x{+}2\ \ 1\ \ 2^s)$ *$(x\neq-3)$*
- $(-3\ \ {-2}^s\ \ \mathbf{-1}\ \ {-2}^t) \rightarrow (1\ \ 2^s\ \ 3\ \ 2^t)$
- $\mathbf{(-2^t)} \rightarrow (2^t)$



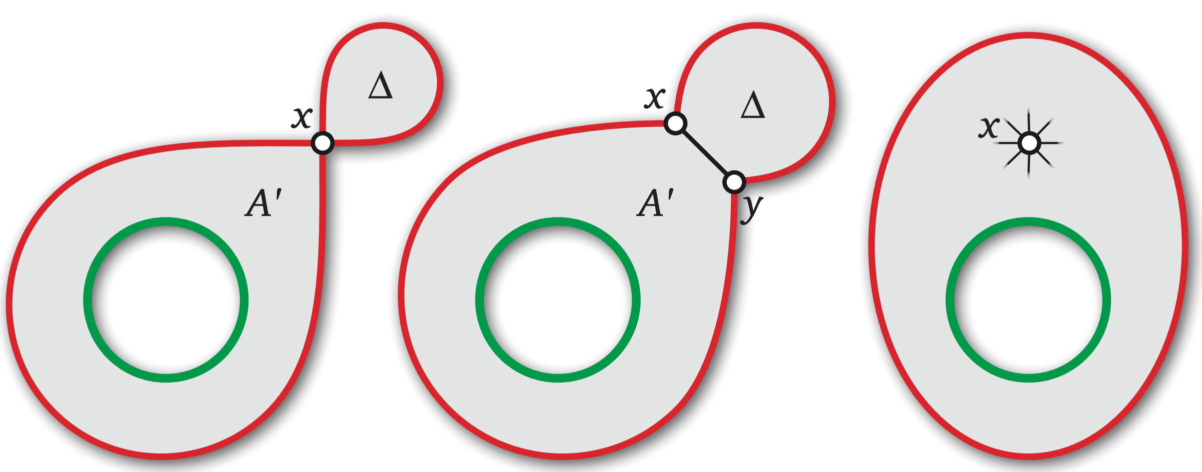
Homotopy algorithm

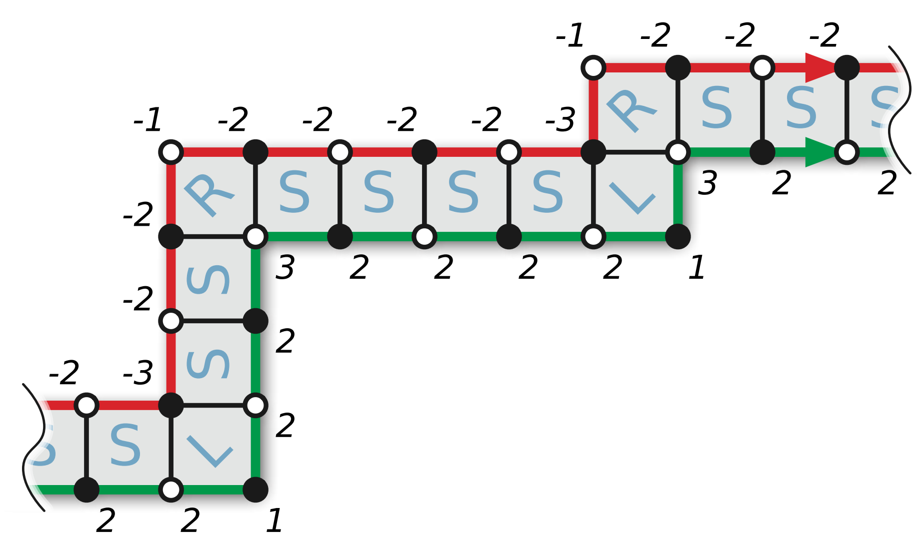
- Reduce both cycles: $O(\ell)$ time
- Make reduced cycles canonical via right shifts: $O(\ell)$ time
  - Run length encoding $\Rightarrow$ each shift takes $O(1)$ time
  - No backtracking required
- Cycles are homotopic iff same canonical cycle: $O(\ell)$ time

Correctness

[Lazarus Rivaud 2012]
[Erickson Whittlesey 2013]

- **Lemma:** *Each free homotopy class contains exactly one canonical cycle.*
- **Corollary:** *Two cycles are freely homotopic iff they yield the same canonical cycle.*
- Proof uses Combinatorial Gauss-Bonnet Theorem again





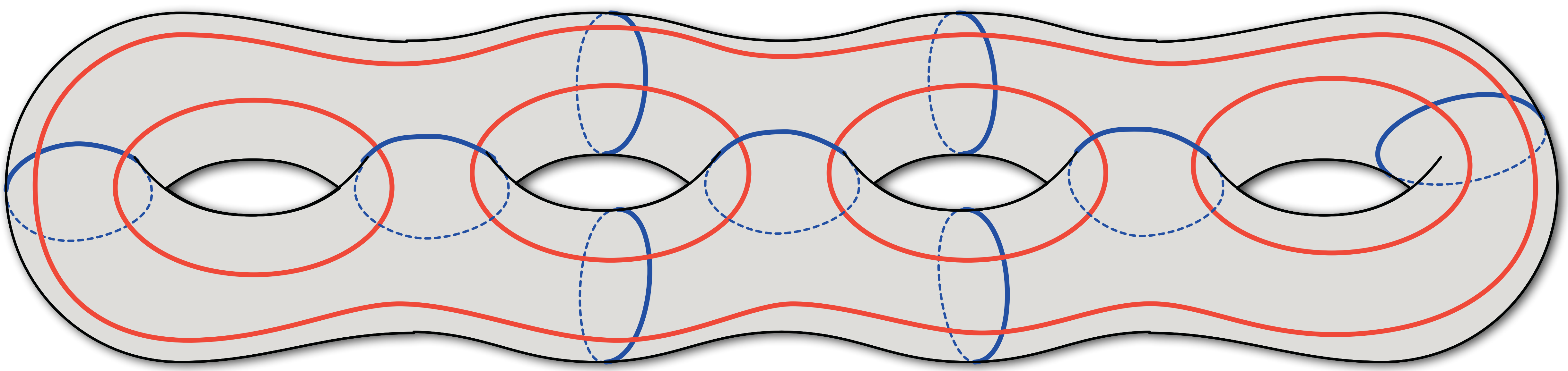
Shortest homotopic paths

Shortest homotopic curves

- Given an *arbitrary* surface map Σ *with weighted edges* and a walk α in Σ, find the shortest walk homotopic to α.
  - Algorithms we've just seen solve this problem when Σ is an *unweighted* system of loops or system of quads

Tight hexagonal decomposition

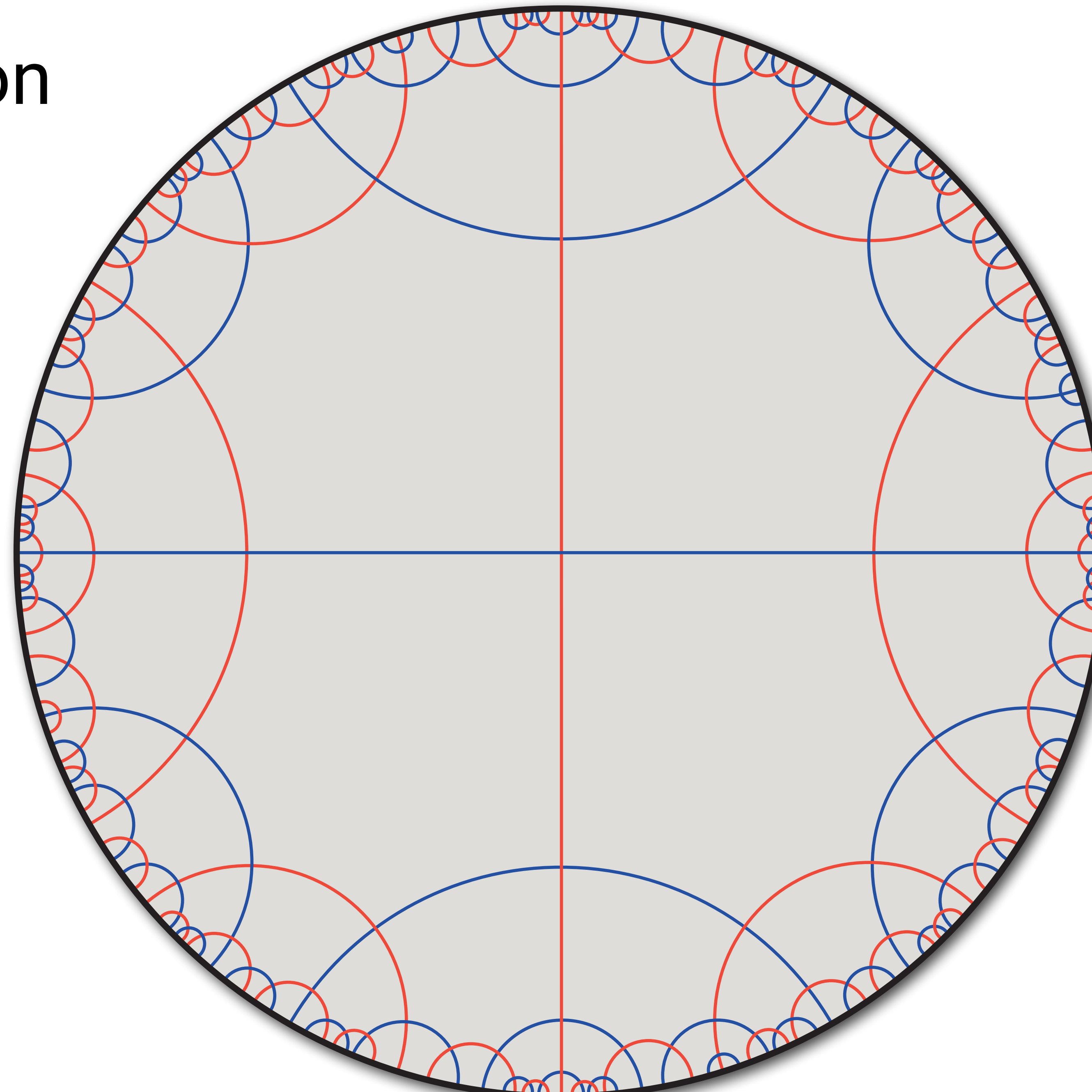
$4g$ cycles, each as short as possible in its homotopy class, that decompose Σ into “right-angled hexagons”



[Colin de Verdière, Erickson 2006]

Universal cover

The hexagonal decomposition lifts to a regular tiling of the hyperbolic plane with right-angled hexagons.



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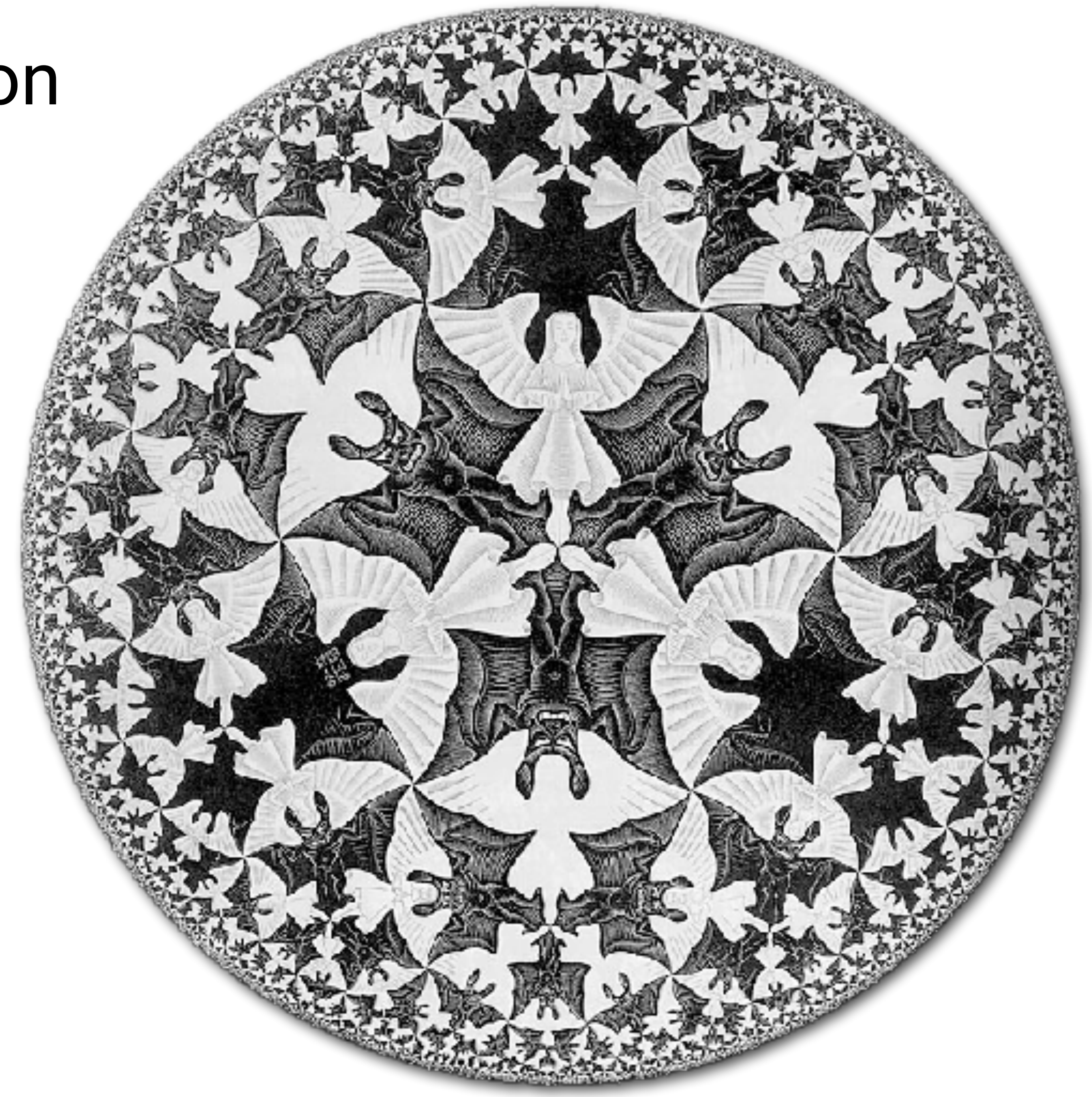

M. C. Escher, *Circle Limit IV: Heaven and Hell* (1960)

Universal cover

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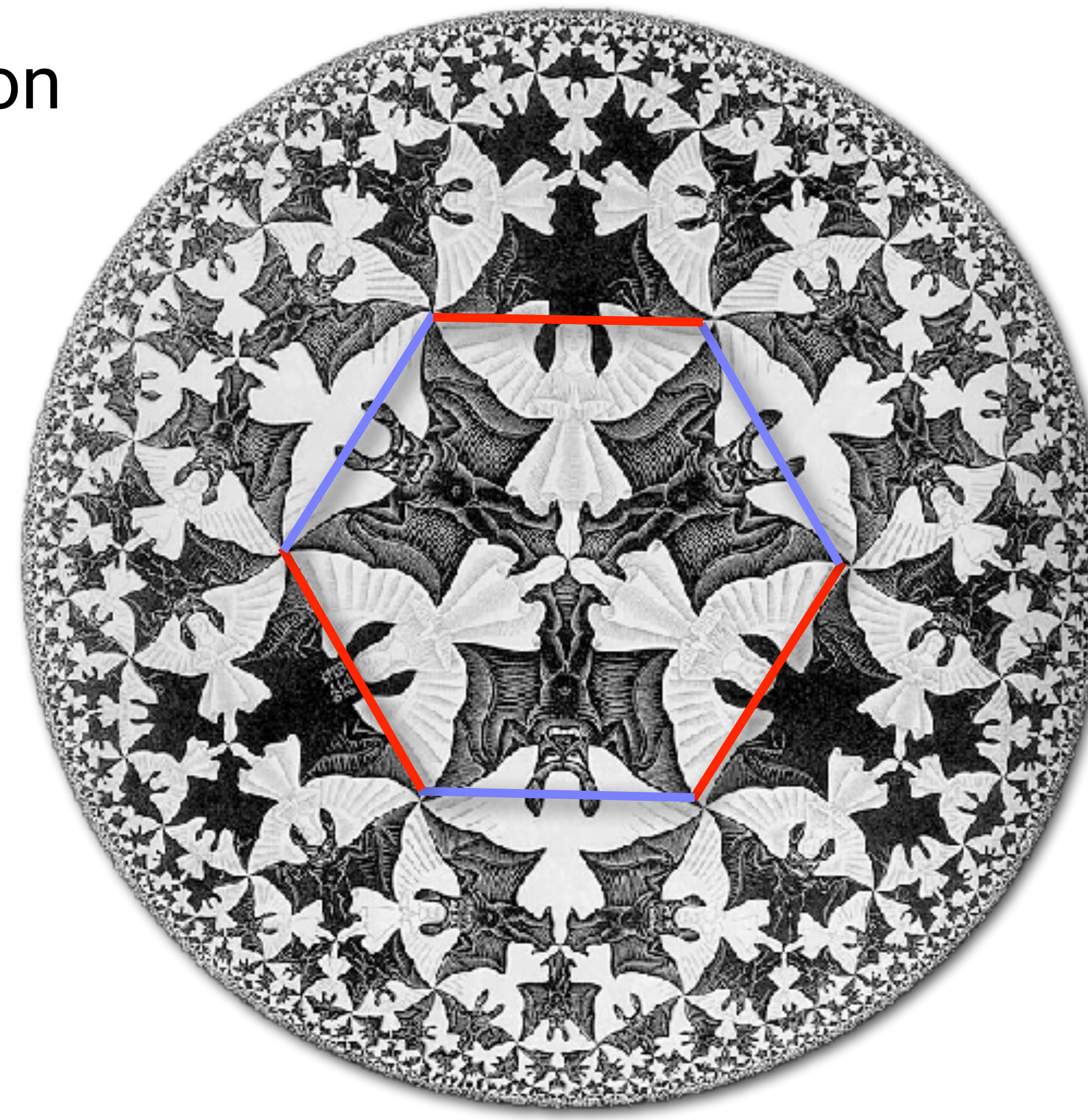
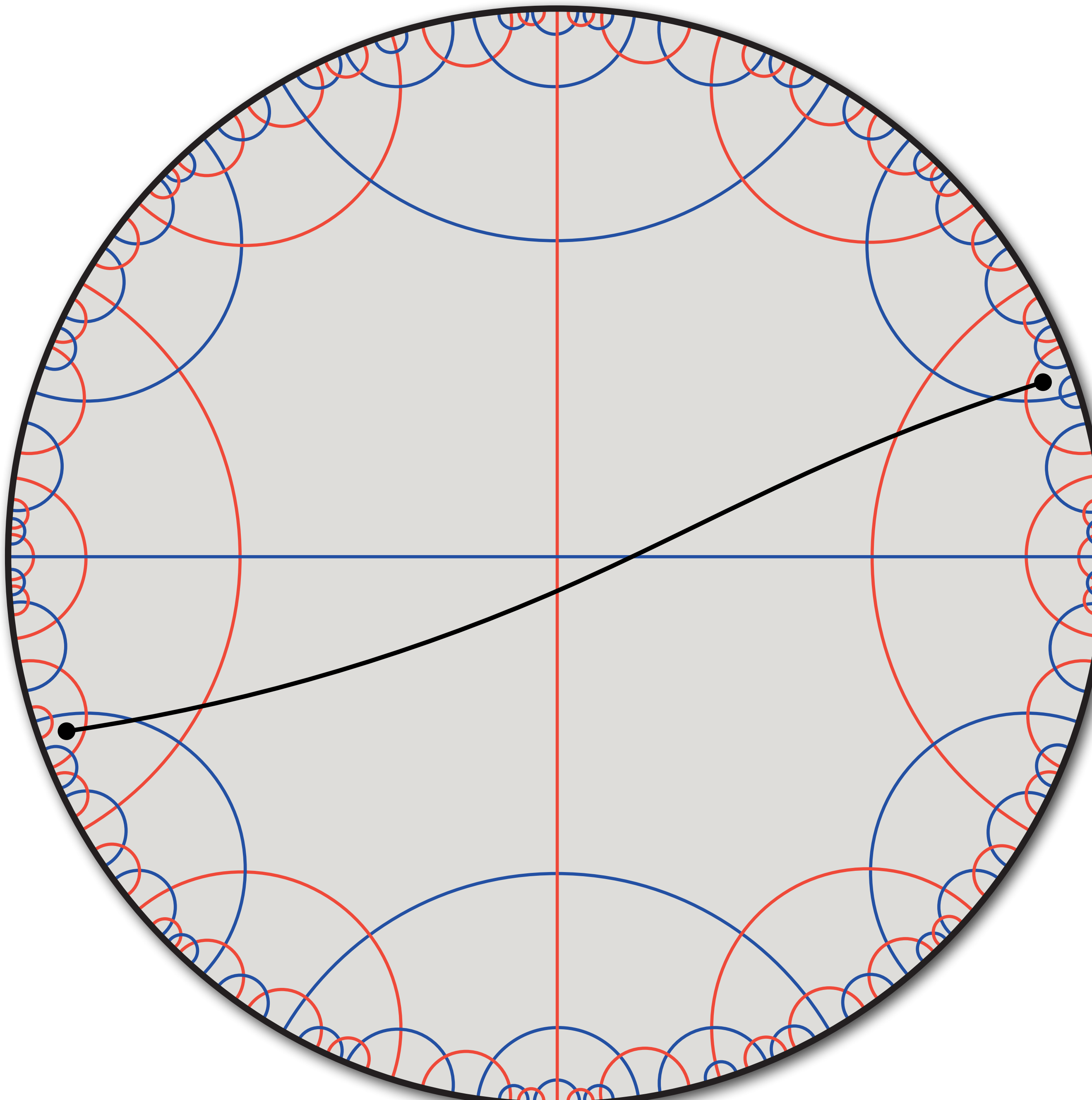


M. C. Escher, *Circle Limit IV: Heaven and Hell* (1960)

Universal cover

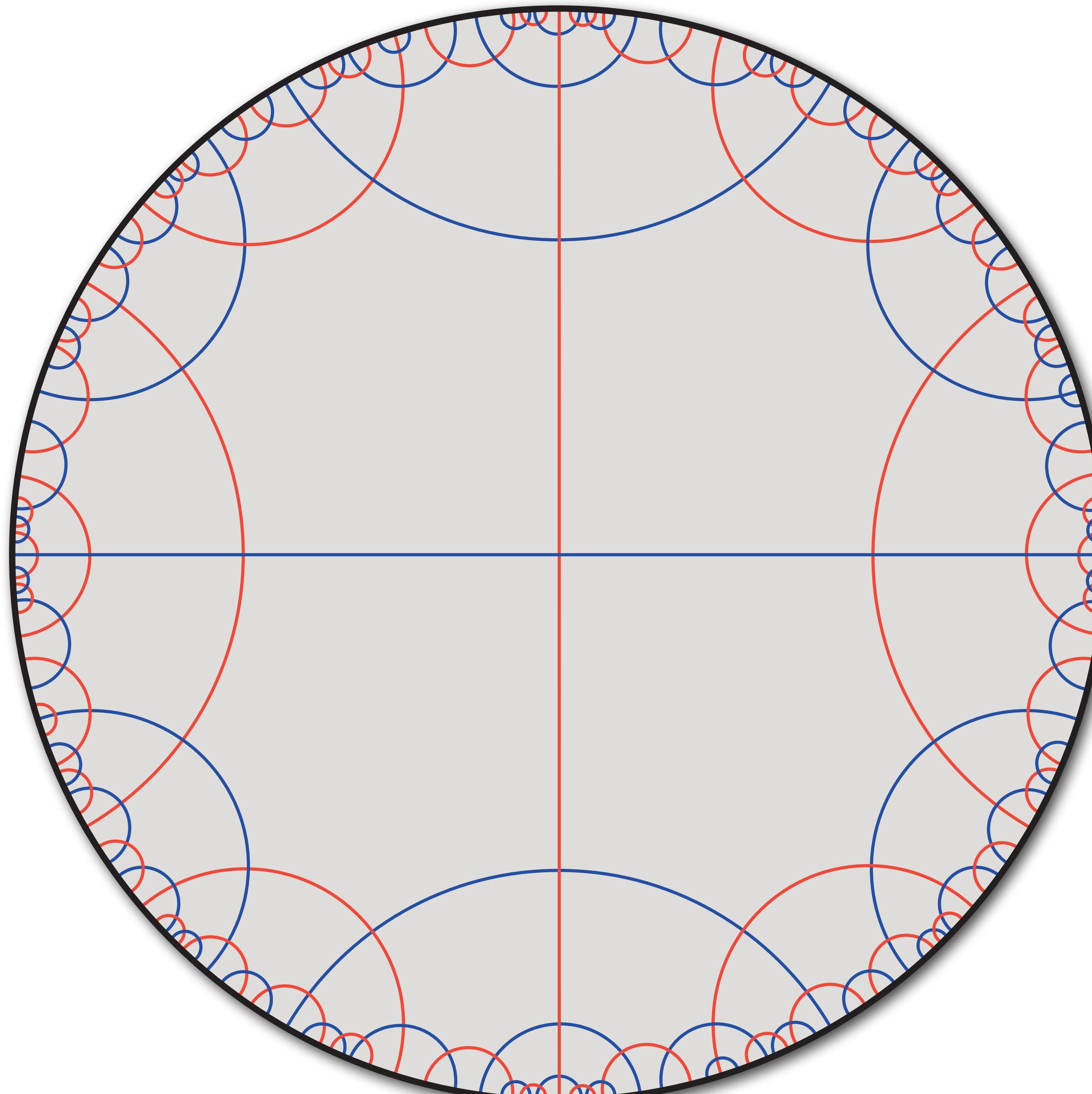
- Each cycle in the tight hexagonal decomposition lifts to a *line*—an infinite shortest path—in $\tilde{\Sigma}$.
- *Shortest* paths in $\tilde{\Sigma}$ cross each line at most once.



[Colin de Verdière, Erickson 2006]

Universal cover

- First encode the input path by the sequence of lines that it crosses.
- Then reduce the crossing sequence using small cancellation.
- The reduced crossing sequence lists the lines that $\tilde{\pi}$ crosses an odd number of times.



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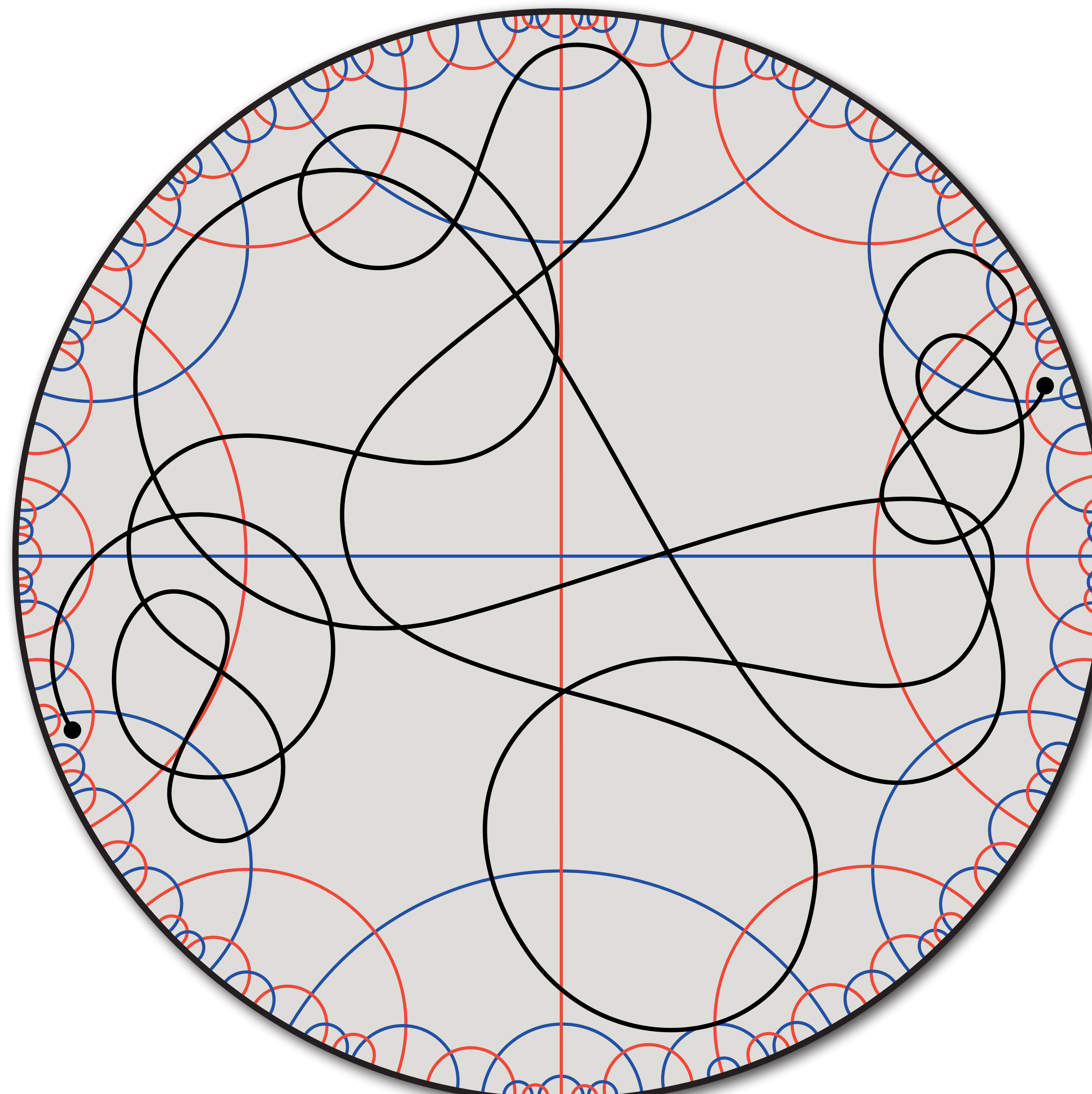
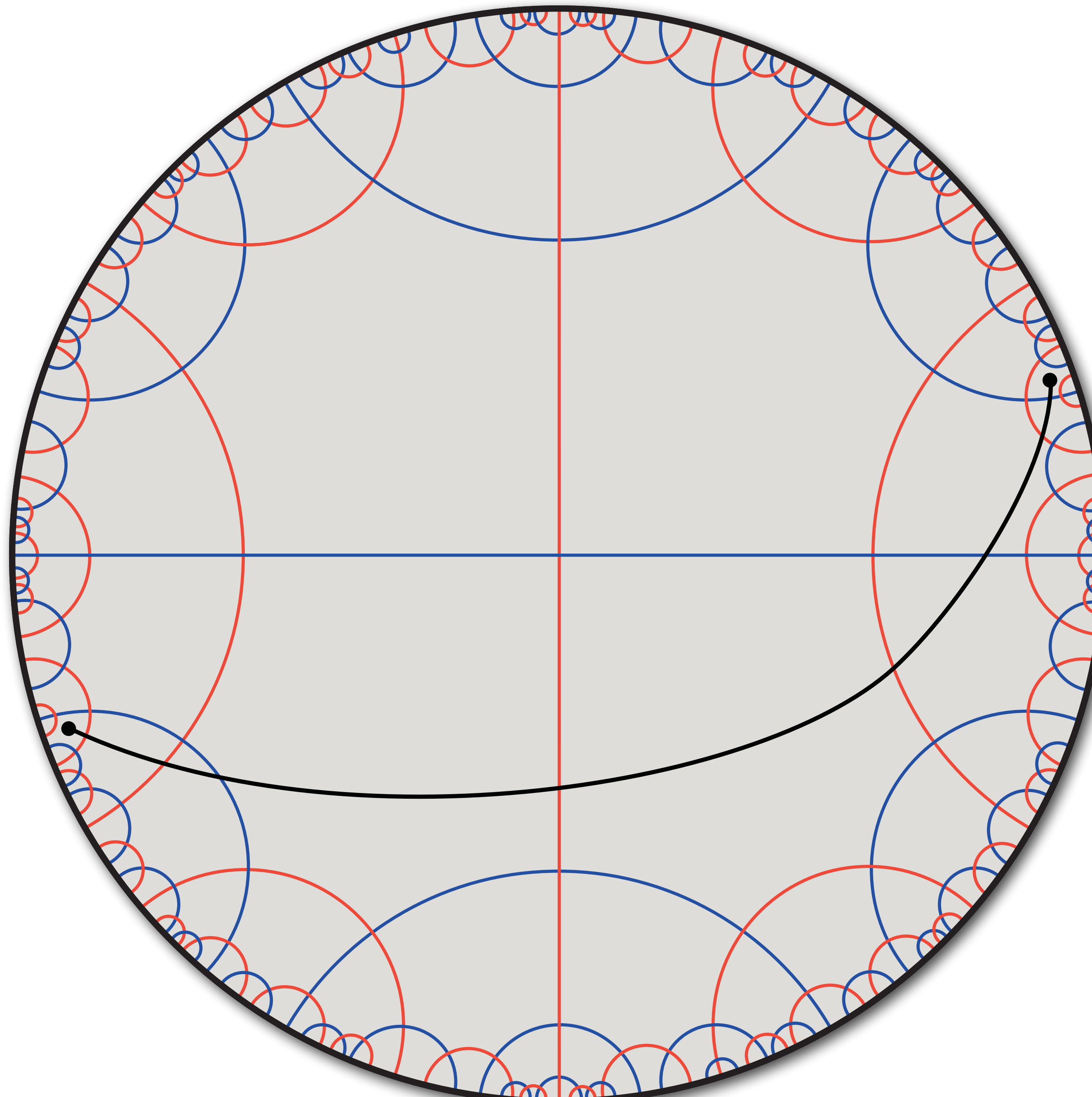


[Colin de Verdière, Erickson 2006]

Universal cover

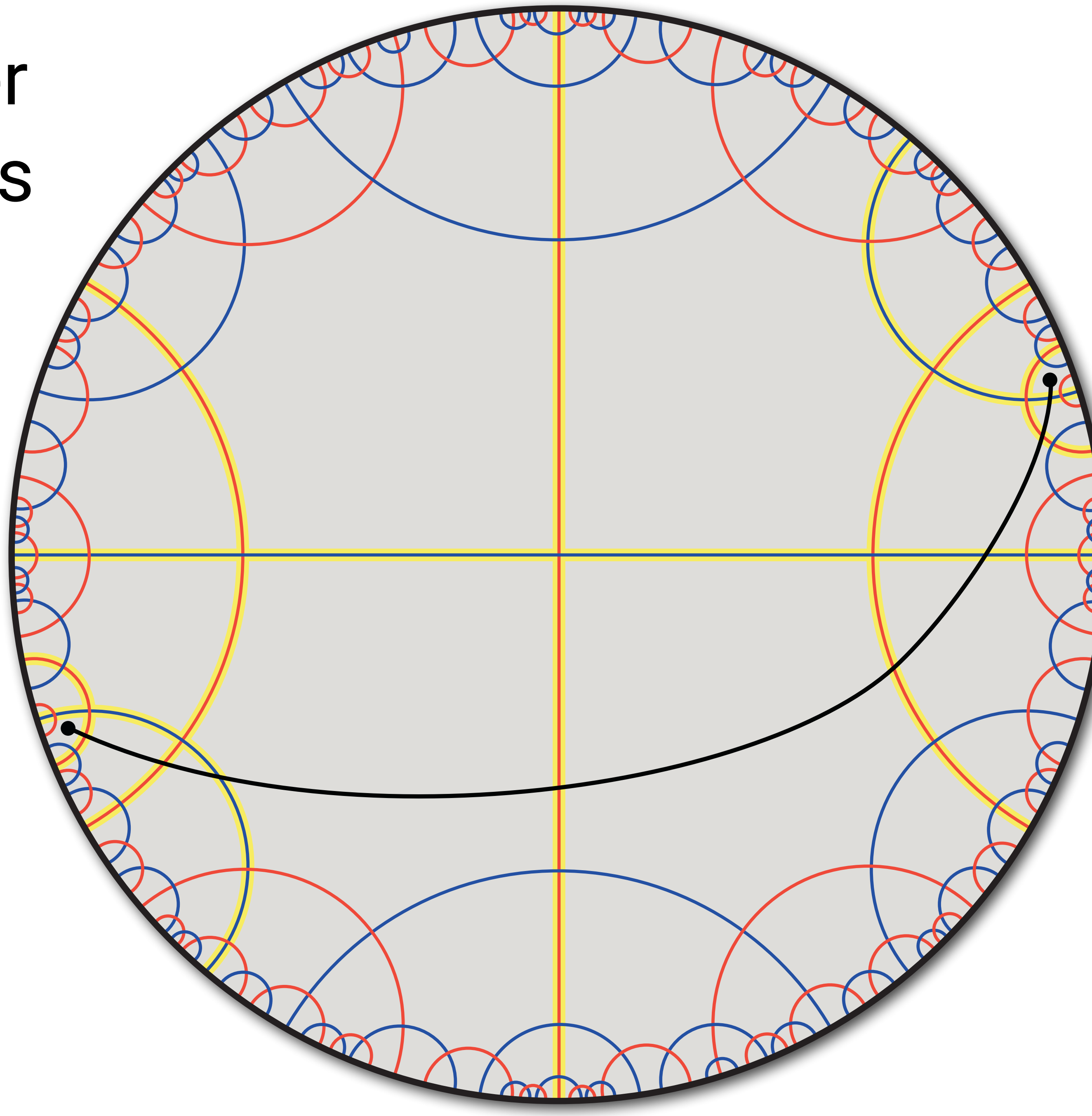
- First encode the input path by the sequence of lines that it crosses.
- Then reduce the crossing sequence using small cancellation.
- The reduced crossing sequence lists the lines that $\tilde{\pi}$ crosses an odd number of times.



[Colin de Verdière, Erickson 2006]

Relevant region

- Hexagons in universal cover containing all reduced paths between endpoints of $\tilde{\pi}$.
- Convex hyperbolic polygon
- Discrete Gauss-Bonnet ⇒ $O(x)$ relevant hexagons
- Comput from the reduced crossing sequence in O(x) time.



[Colin de Verdière, Erickson 2006]

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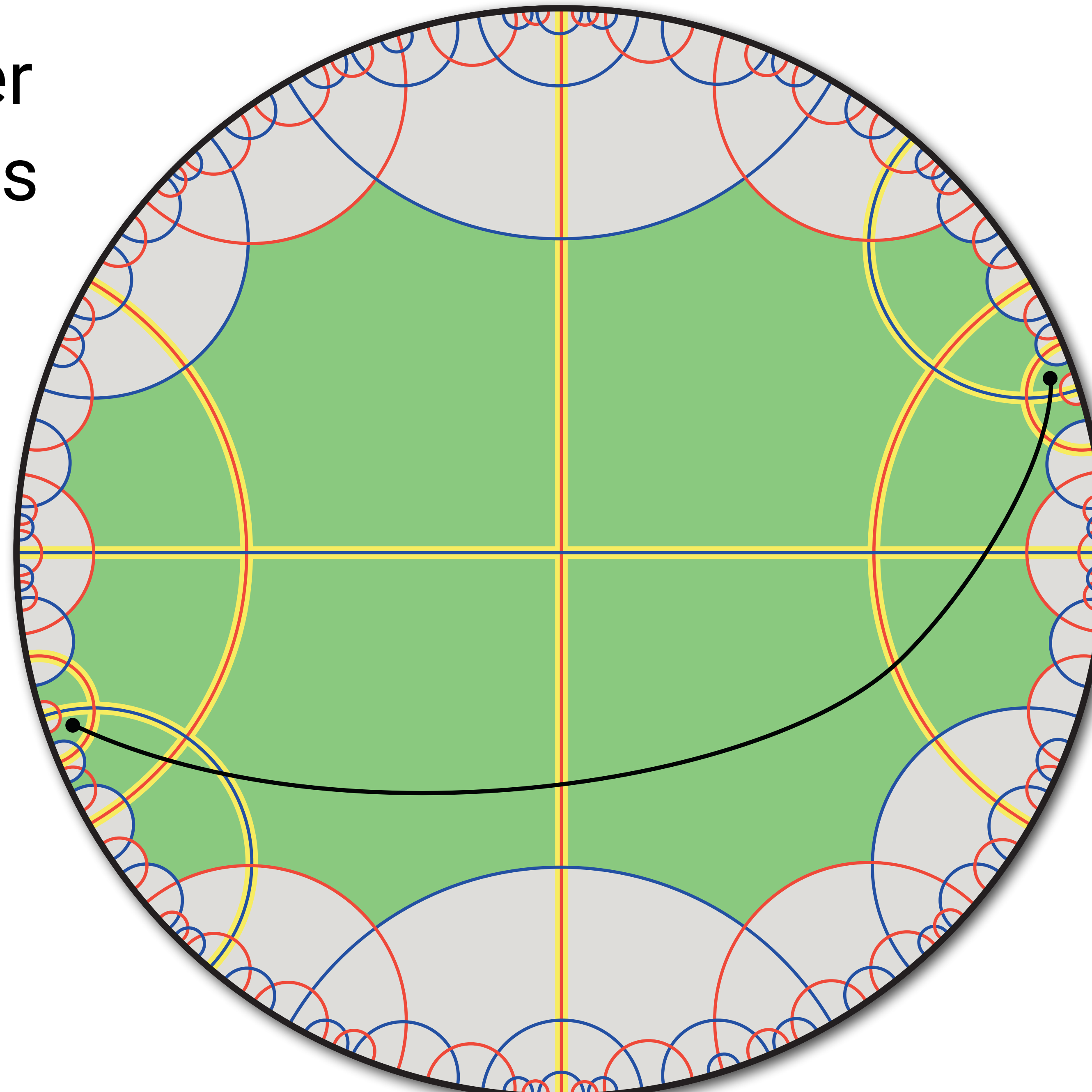

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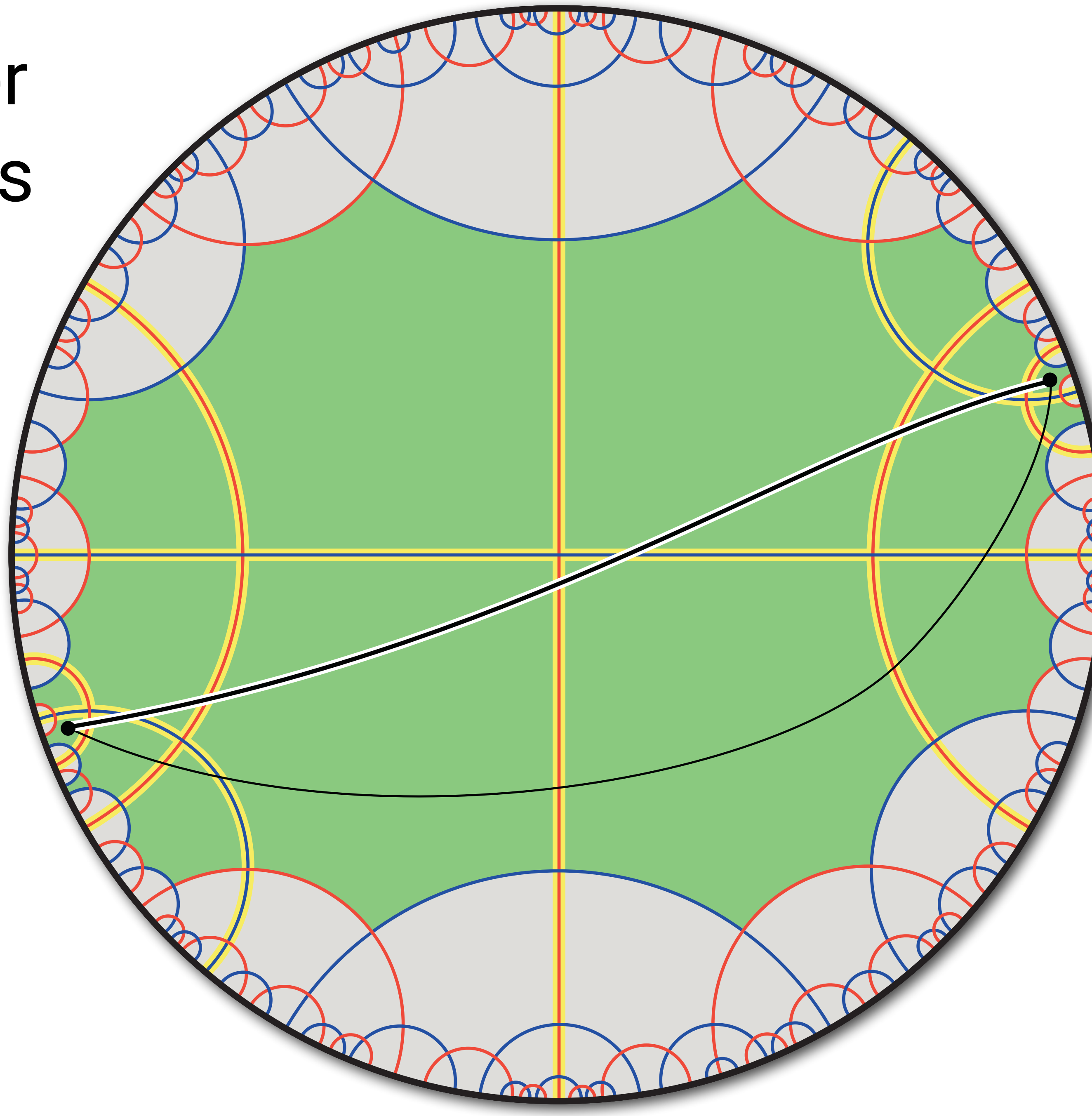

[Colin de Verdière, Erickson 2006]

Shortest homotopic paths

- Given a path π with ℓ edges in a surface map, the shortest path homotopic to π can be found in $O(gn \log n + gn\ell)$ time.
  - Build a tight hexagonal decomposition — $O(gn \log n)$ time
  - Compute the crossing sequence of π — $O(x) = O(g\ell)$ time
  - Reduce crossing sequence via small cancellation — $O(x)$ time
  - Build relevant region of $\tilde{\Sigma}$ — $O(xn)$ time
  - Find shortest path in relevant region — $O(xn)$ time

[Colin de Verdière, Erickson 2006]

Higher dimensions?

Undecidability

- Contractibility and homotopy equivalence of curves in non-manifold 2-complexes and 4-manifolds is *undecidable*!
  - Contractibility = *word problem* in the fundamental group.
    Homotopy = *conjugacy problem* in the fundamental group.
    [Dehn 1911]
  - The word and conjugacy problems are *undecidable* for arbitrary finitely presented groups.
    [Novikov 1954–55] [Boone 1959]
  - For any finitely presented group G, there is a 2-complex/4-manifold whose fundamenal group is G.
    [Markov 1958]

3-Manifolds

- Homotopy is decidable, but no explicit time bounds
  [Perelman 2003] [Aschenbrenner Friedl Wilton 2015]
  - "Brute force" algorithm takes triply exponential time
    [Epstein et al. 1992]
- Contractibility of a *simple boundary* curve is in *NP* (and therefore decidable in exponential time).
  [Kneser 1929] [Haken 1961] [Hass Lagarias Pippenger 1999]
  [Agol Hass Thurston 2006]
- Contractibility of an *arbitrary boundary* curve is decidable in *exponential time* (but not known to be in NP).
  [Colin de Verdière, Parsa 2017]

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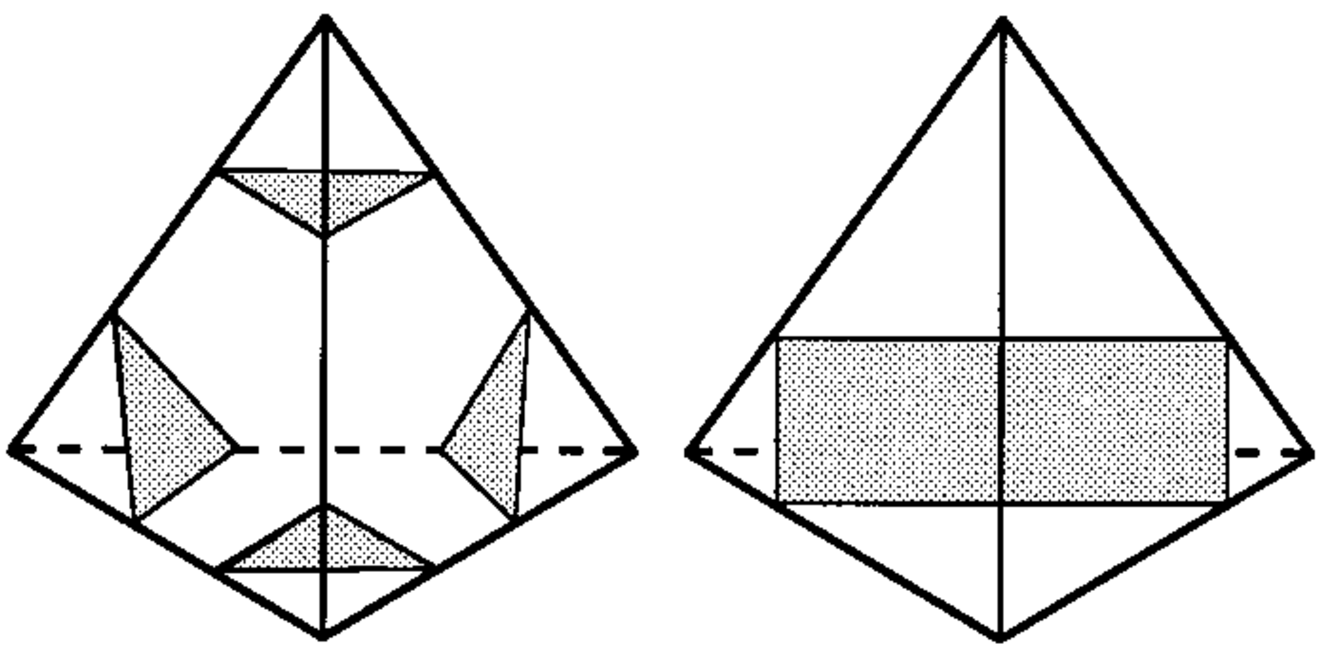

FIG. 1. Elementary disks in a normal surface.

3-Manifolds

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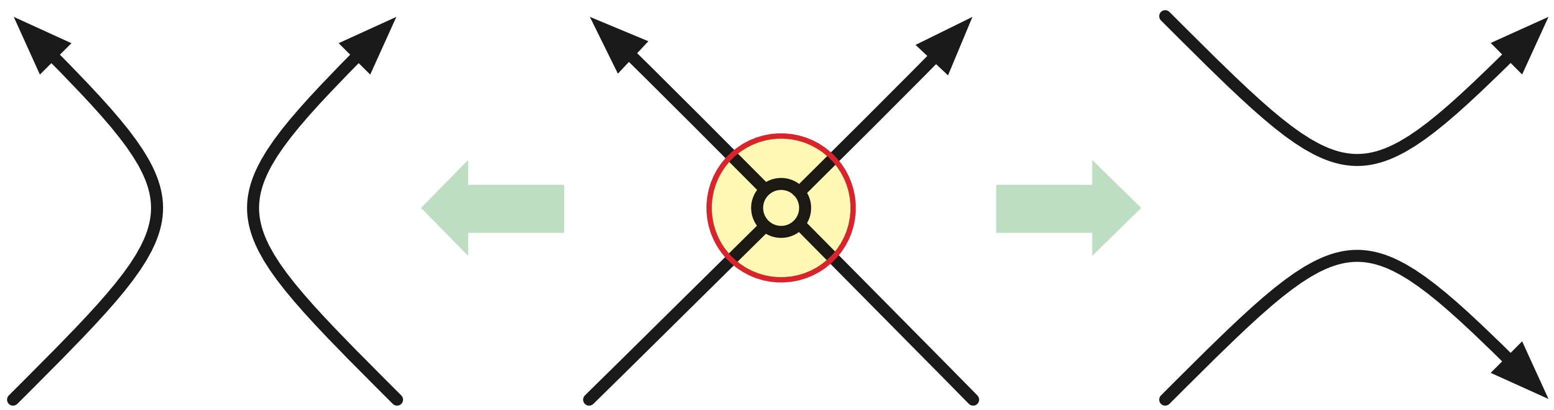
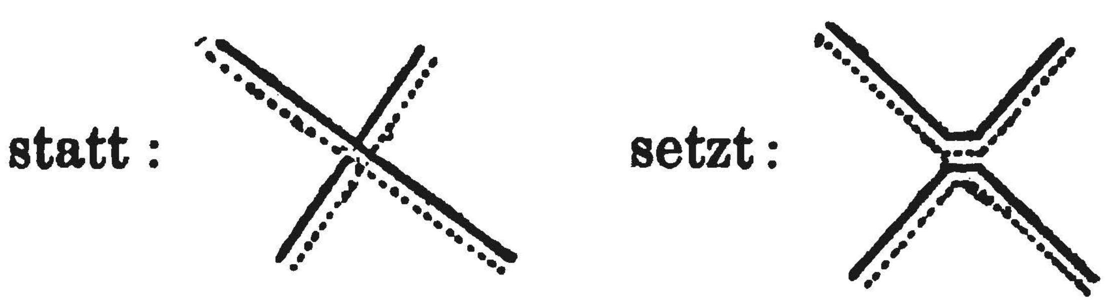
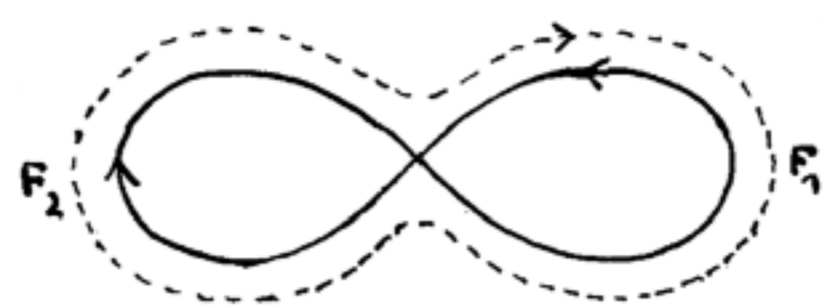
  [Colin de Verdière, Parsa 2017]





[Gauss c.1840]



[Dehn 1936]

Thank you!