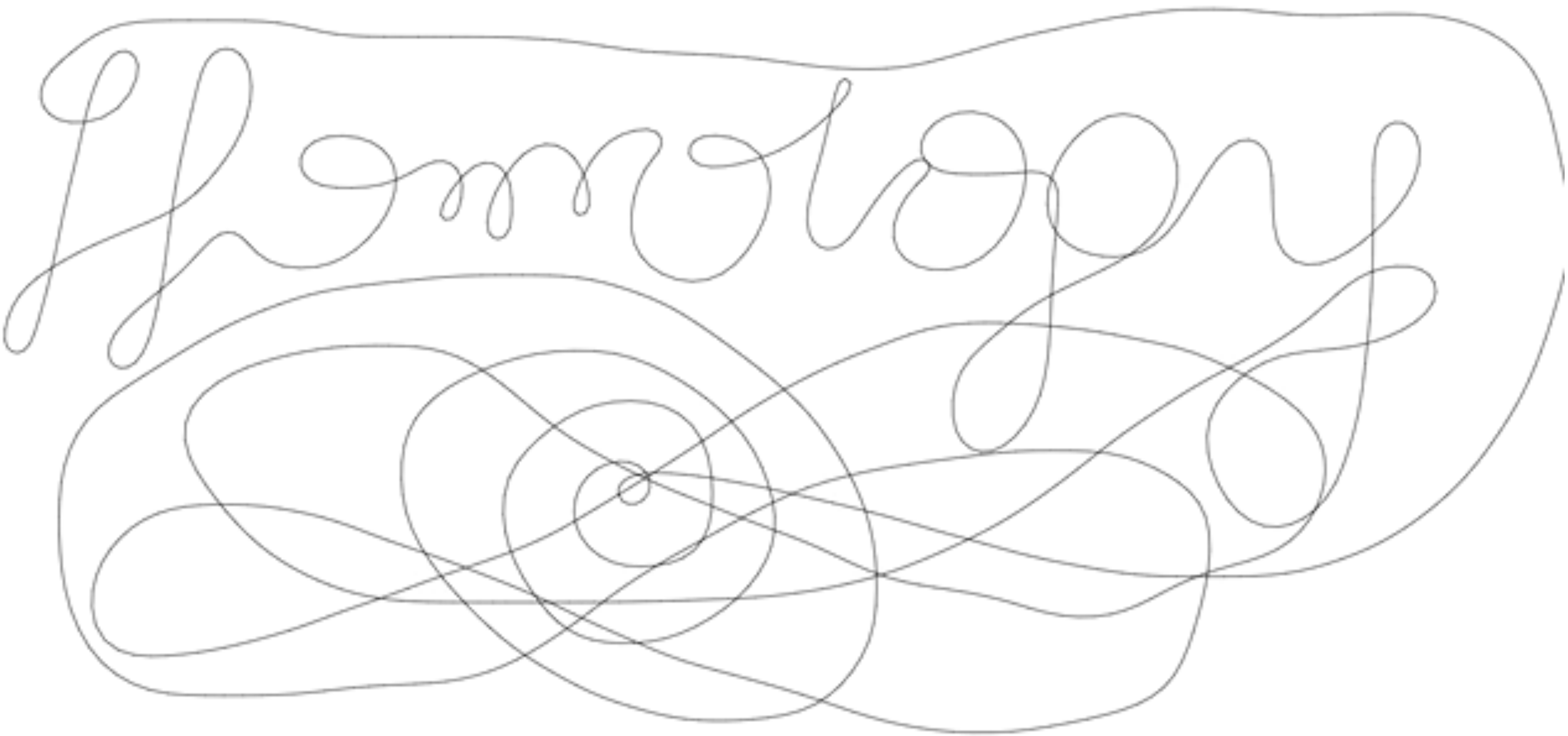
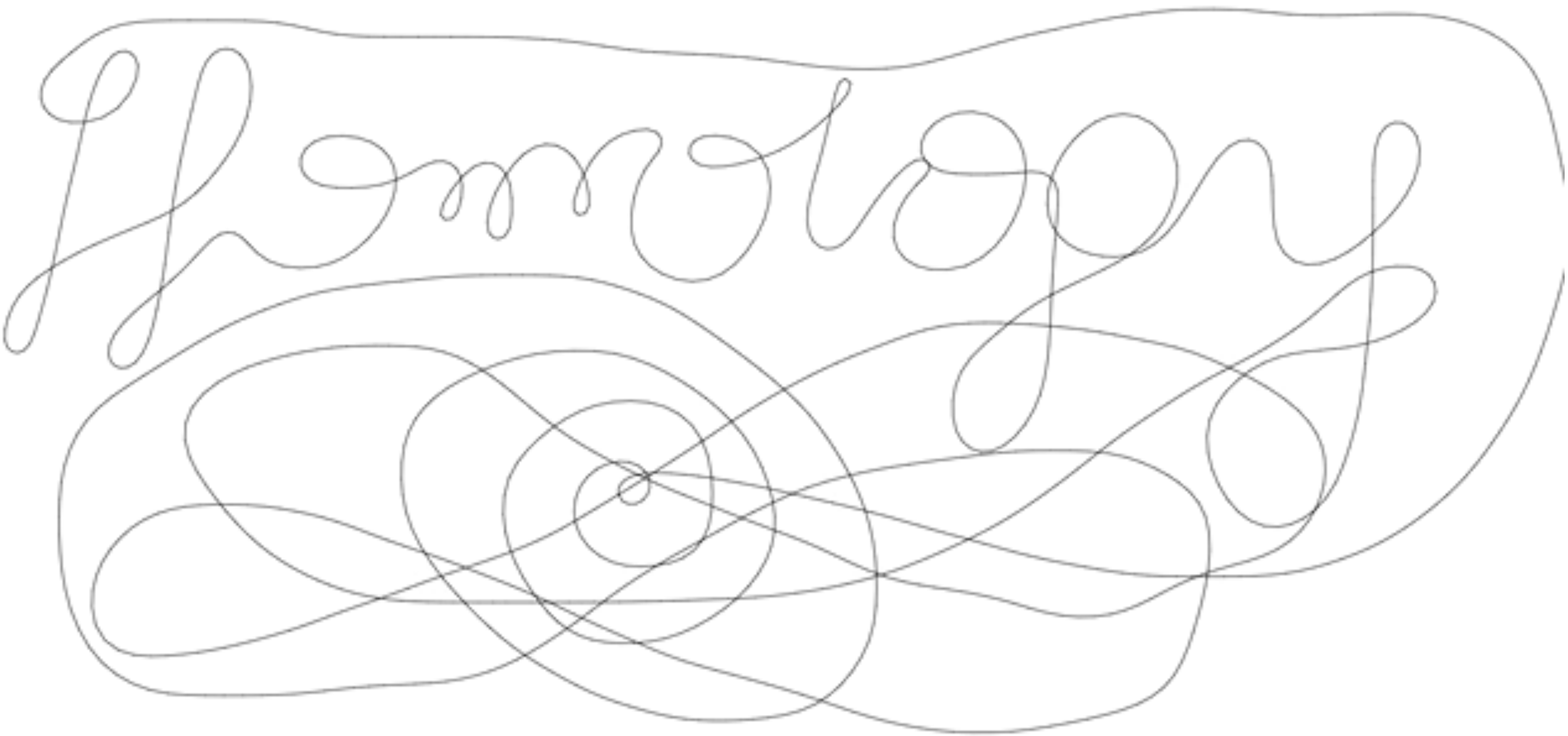
One-Dimensional Computational Topology

V. Untangling and unwinding curves

Jeff Erickson
University of Illinois, Urbana-Champaign


Homology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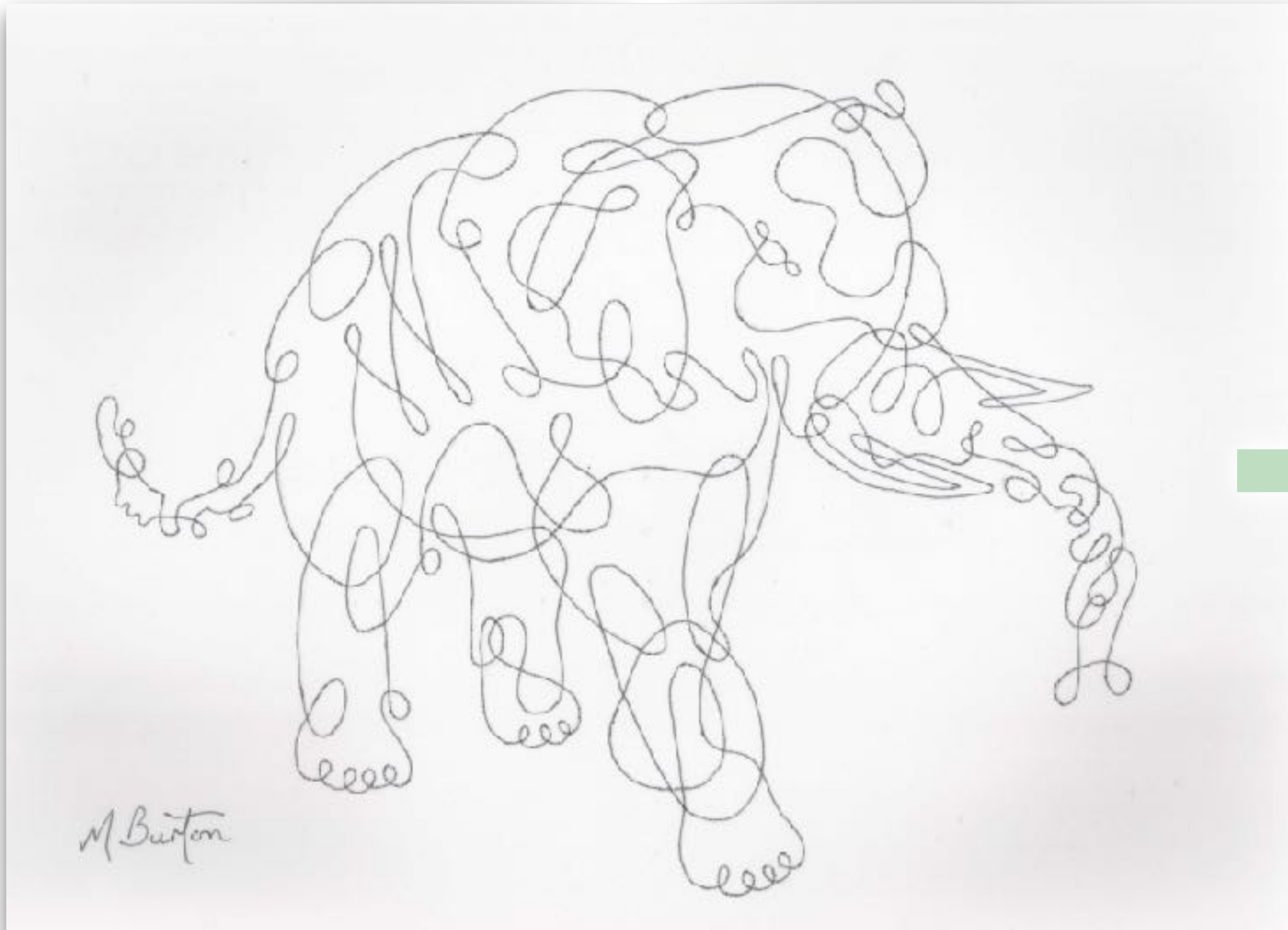
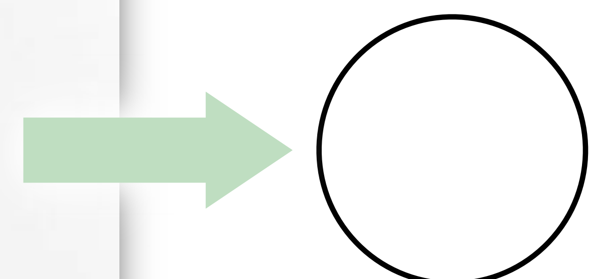
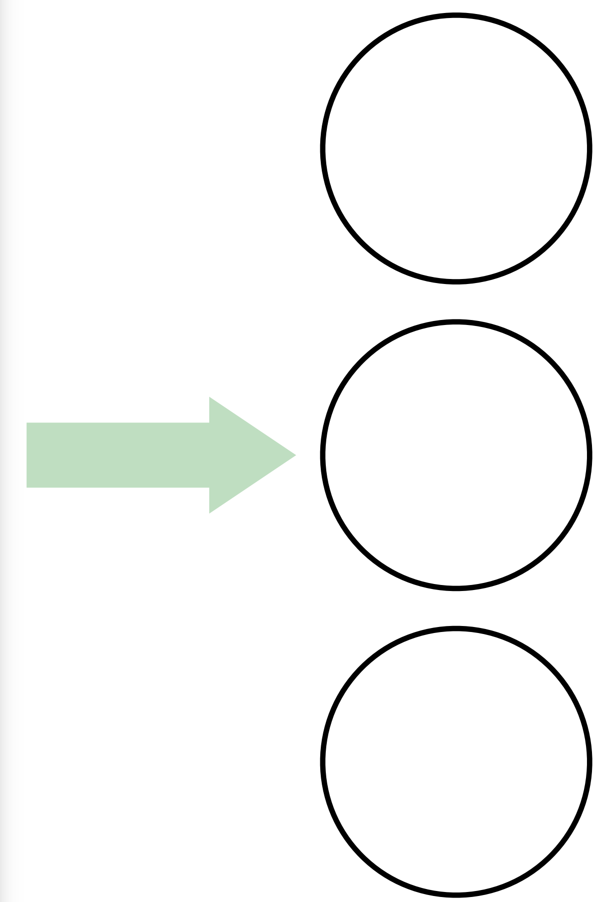


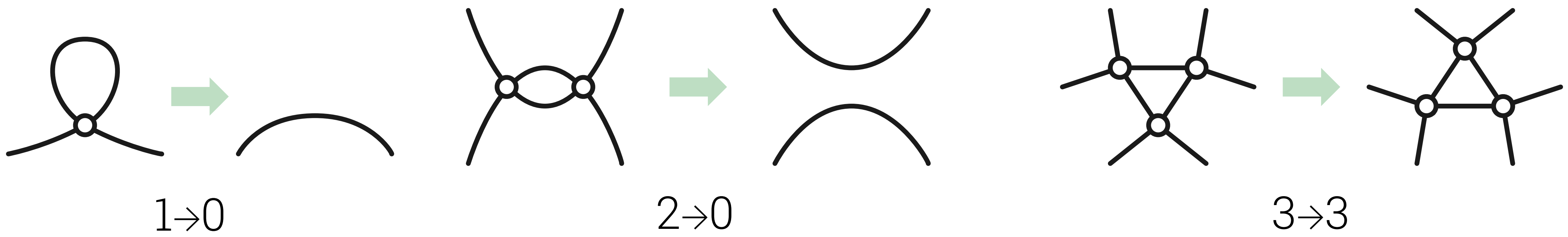


Elephant, Mick Burton (1969)

Der Mann mit dem Mundwerk, Paul Klee (1930)



Homotopy moves



- **Theorem:** Any closed curve in the plane can be simplified using *a finite number of* homotopy moves.
  [Steinitz 1916, Alexander 1926, Alexander Briggs 1927, Reidemeister 1927, Grayson 1989, Angenent 1988, Angenent 1991]

How many?

Previous results

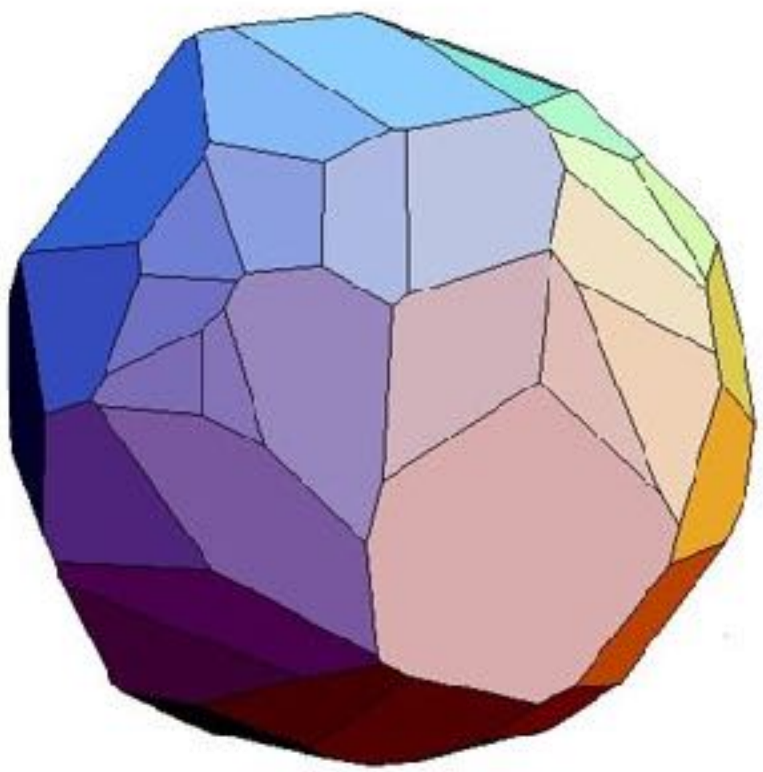
- **$O(n^2)$** homotopy moves are always sufficient
  *[**Steinitz 1916**; Grünbaum 1967; Francis 1971; Feo 1985; Truemper 1989; Vegter 1989; **Feo Provan 1993;** Hass and Scott 1994; Nowik 2000; …]*

- **$\Omega(n)$** homotopy moves are sometimes necessary
  [trivial]

Steinitz's Theorem

[Steinitz 1916]

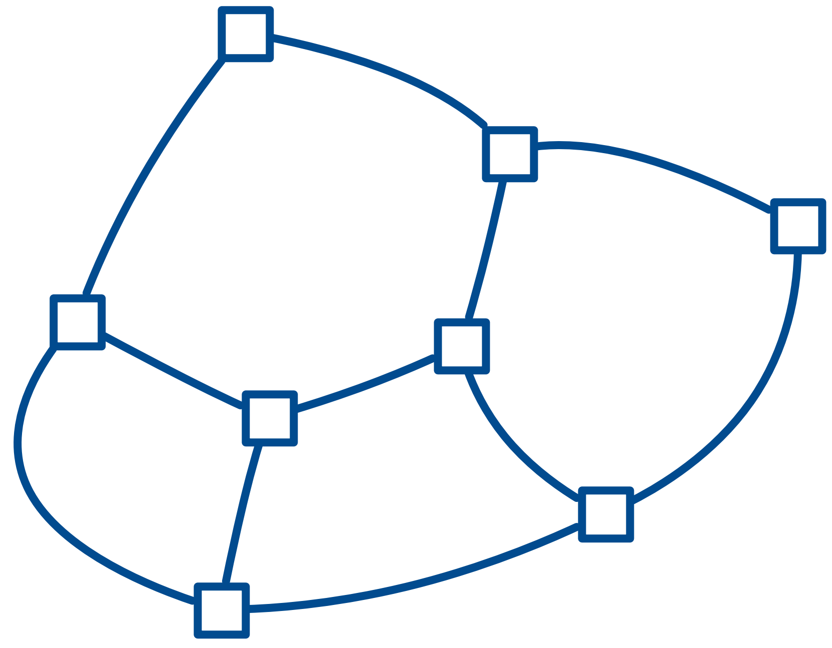
A graph is the 1-skeleton of a convex polytope in $\mathbf{R}^3$ if and only if it is planar and 3-connected.



Medial graph $G^\times$

[Tait 1877, Steinitz 1916]

- Medial vertex for each edge of G
- Medial edge for each corner of G



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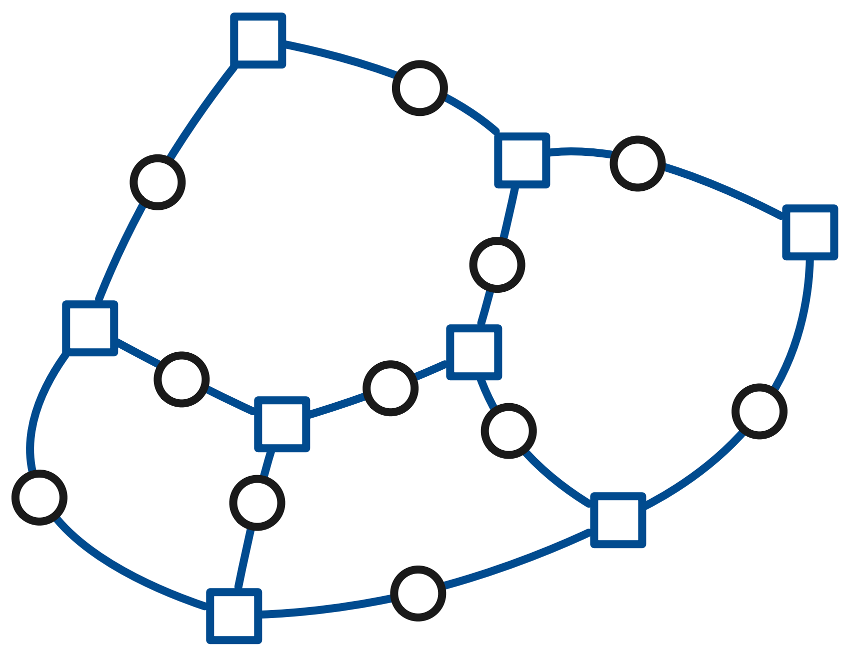

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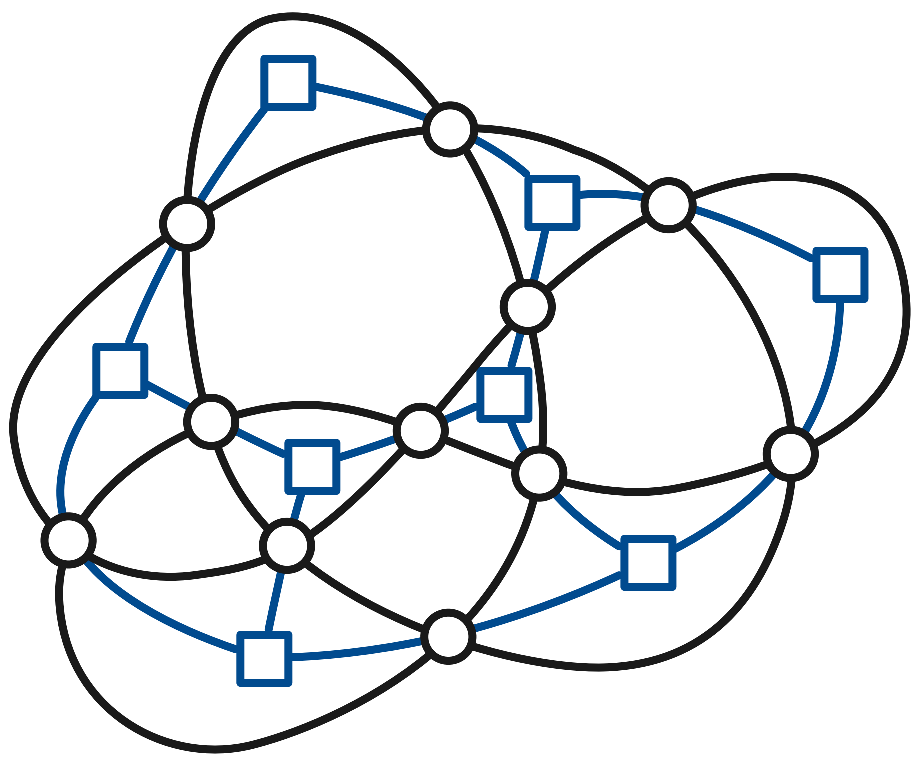

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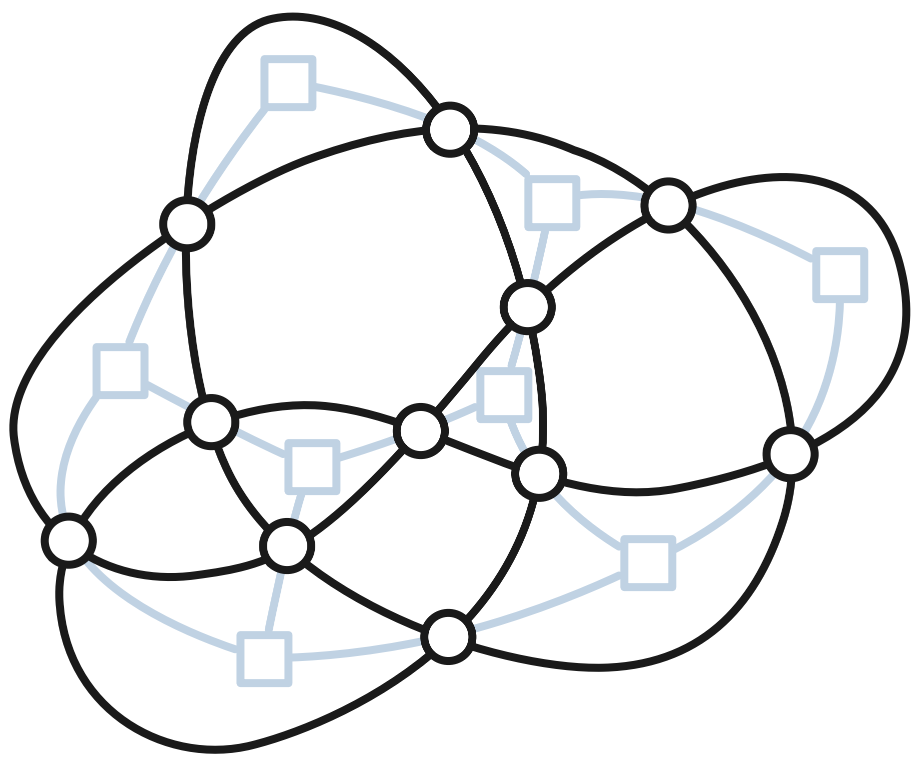
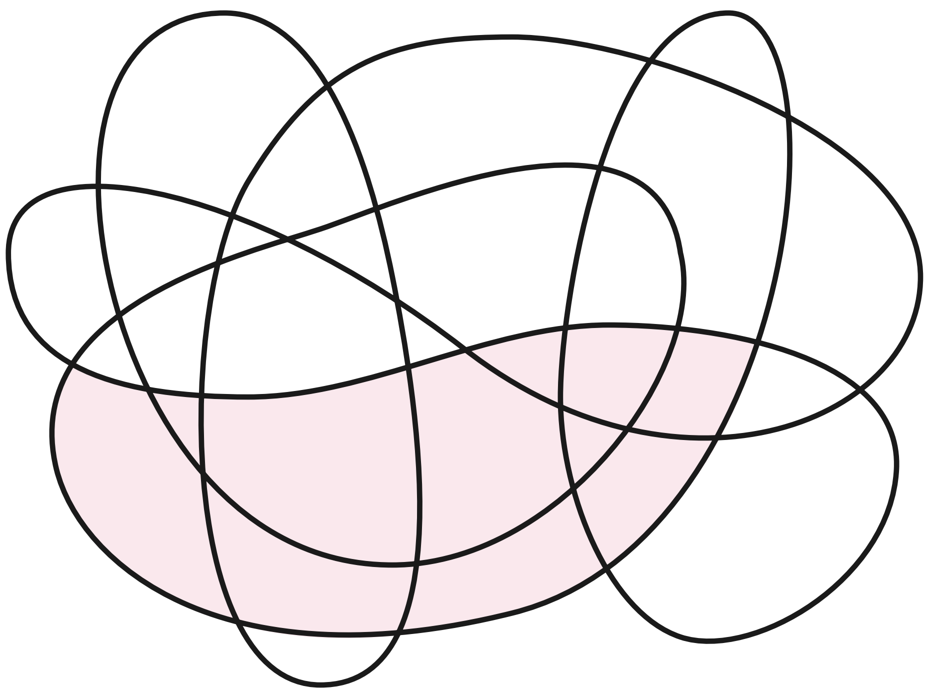


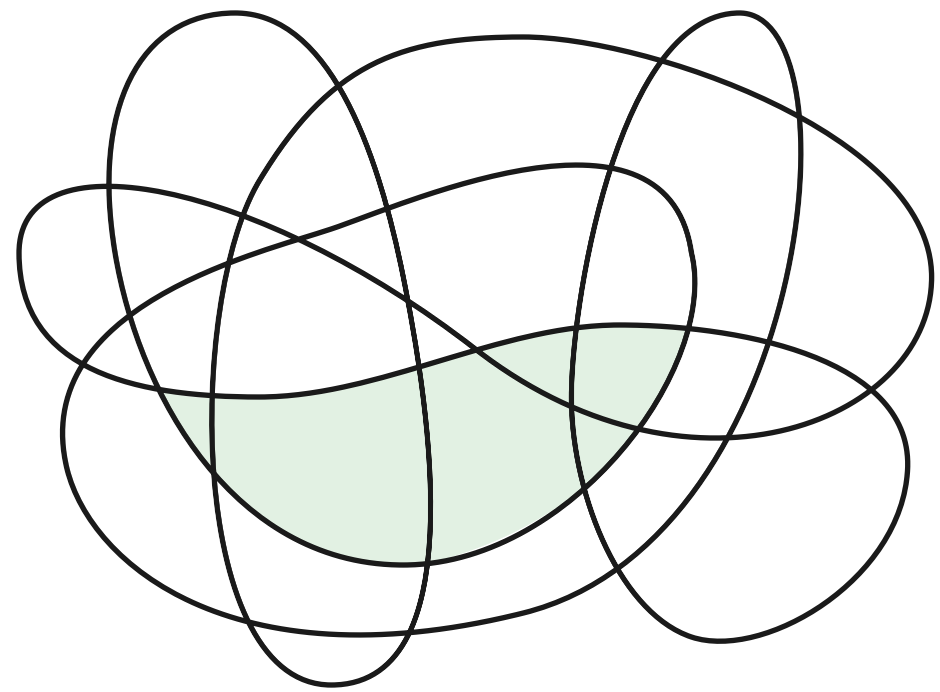
Steinitz's Lemma 1

[Steinitz 1916]

Every 4-regular plane graph contains either an *empty monogon* or a *minimal bigon*.



bigon



minimal bigon

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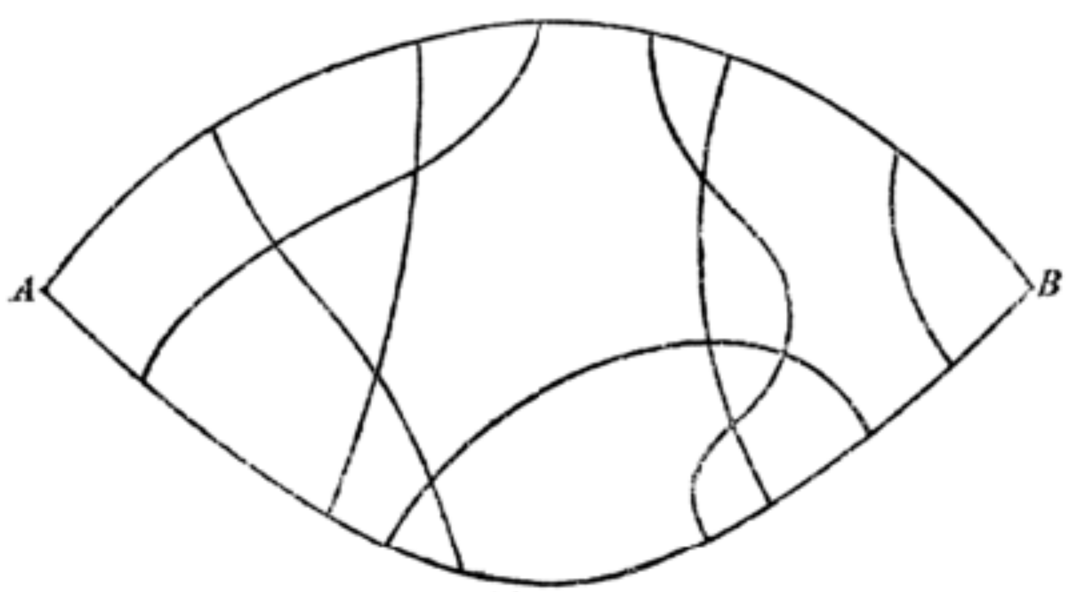


Fig. 21.

Spindel

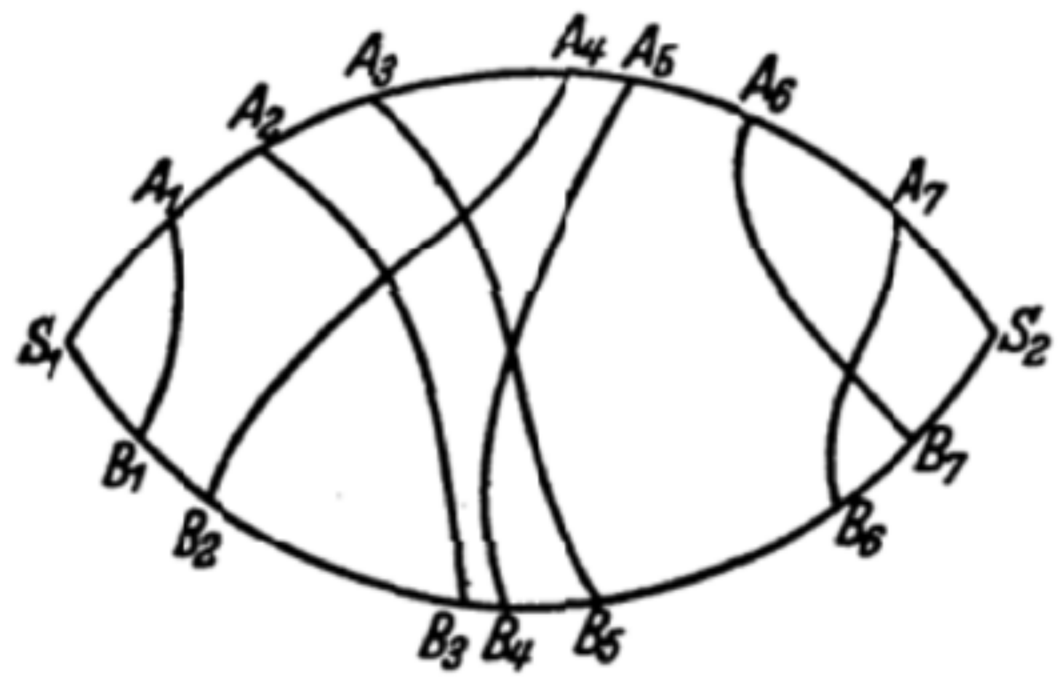


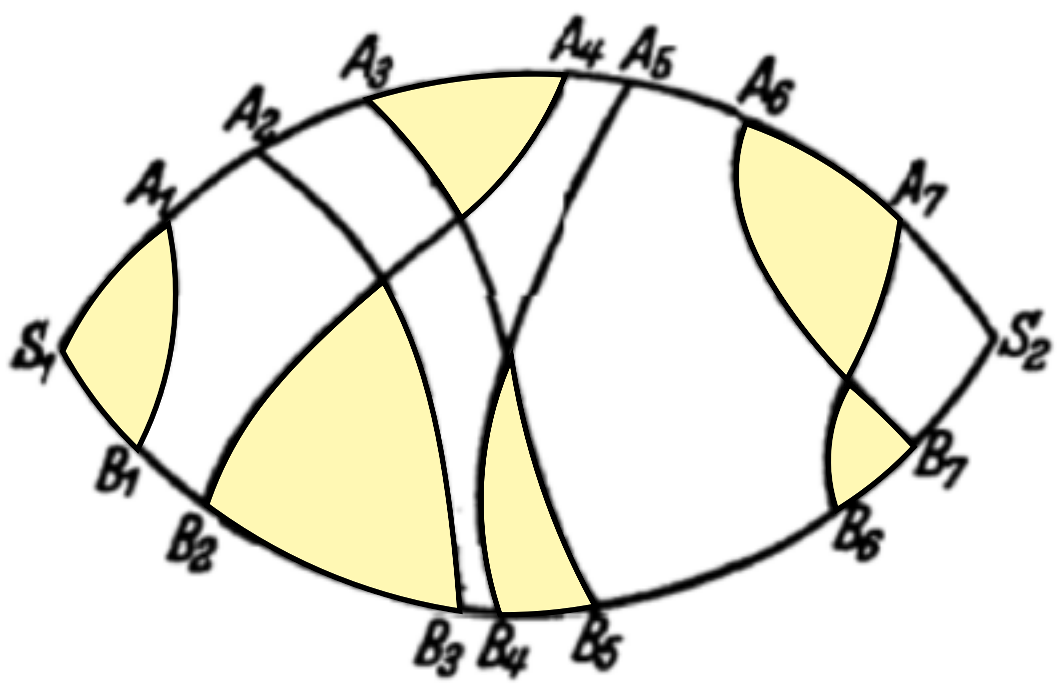
Abb. 175.

irreduzible Spindel

Steinitz's Lemma 2

[Steinitz 1916]

Every non-empty minimal bigon contains *at least two triangular faces* (one adjacent to each side).



Steinitz’s Lemma 2

[Steinitz 1916]

Every non-empty minimal bigon contains *at least two triangular faces* (one adjacent to each side).

So we can reduce any minimal bigon to an *empty* bigon using 3→3 moves

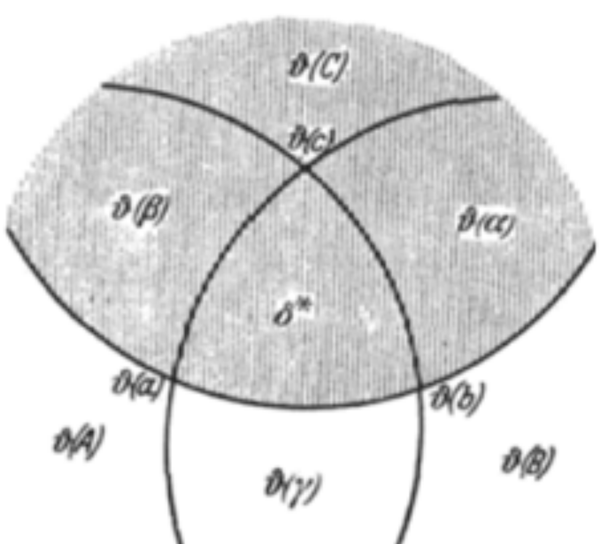


Abb. 176.

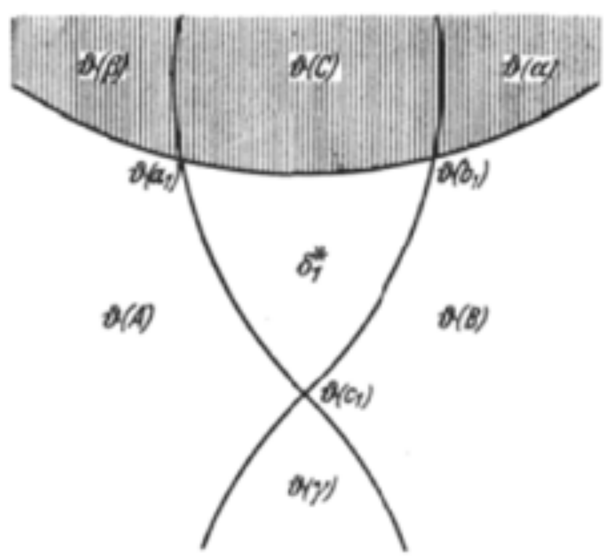


Abb. 178.

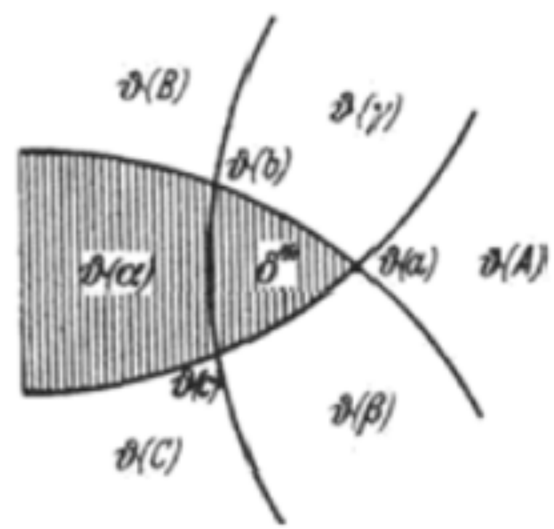


Abb. 179.

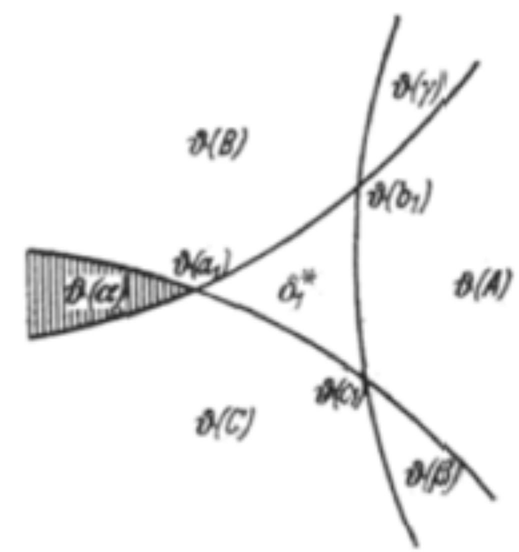
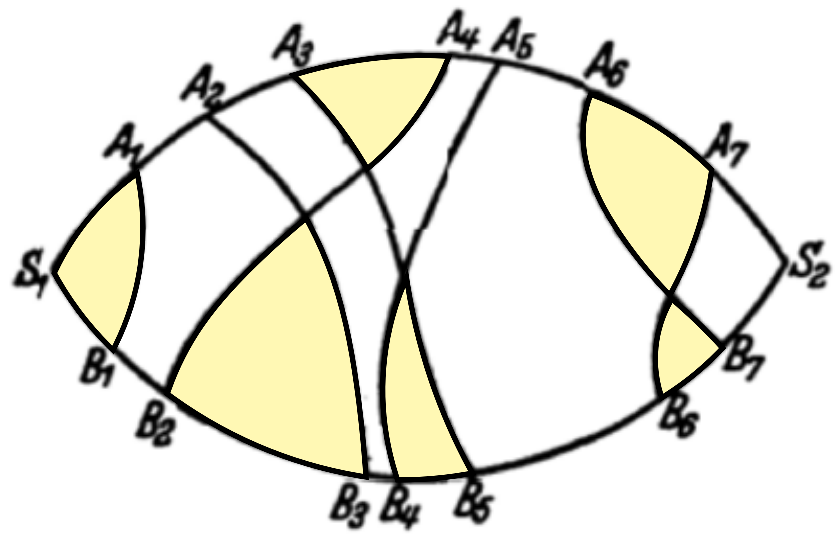


Abb. 180.

Steinitz's Algorithm

[Steinitz 1916]

- While there are vertices
  - If there is an empty monogon, remove it with a 1→0 move
  - Otherwise, **empty** any minimal bigon with 3→3 moves, and then **remove** it with a 2→0 move



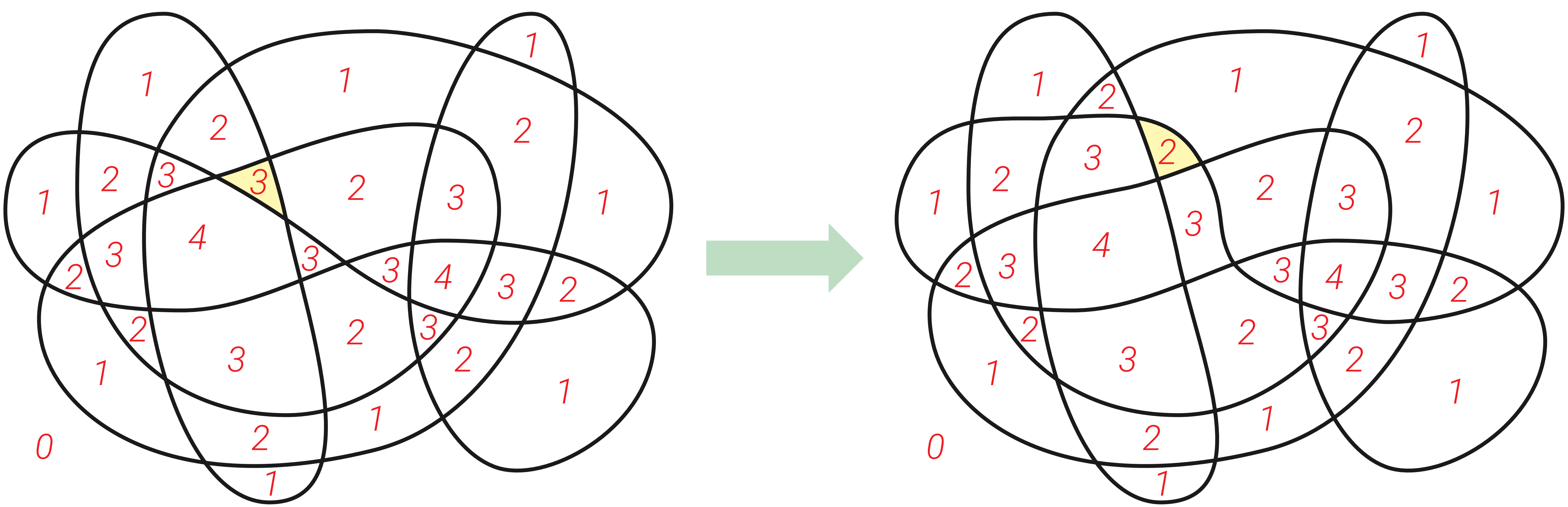
- O(n) moves per bigon = O(n^2) moves

Positive 3→3 moves

[Feo Provan 1993]

The *potential* of a curve is the sum of its face depths.

If a curve has no empty mongons or empty bigons, then some 3→3 move *decreases its potential*.



Feo and Provan's Algorithm

[Feo Provan 1993]

- While there are vertices
  - If there is an empty monogon, perform a 1→0 move
  - Else if there is an empty bigon, perform a 2→0 move
  - Else perform *any* positive 3→3 move

- $O(\Phi) = O(n^2)$ moves, where Φ = potential = sum of face depths

New result

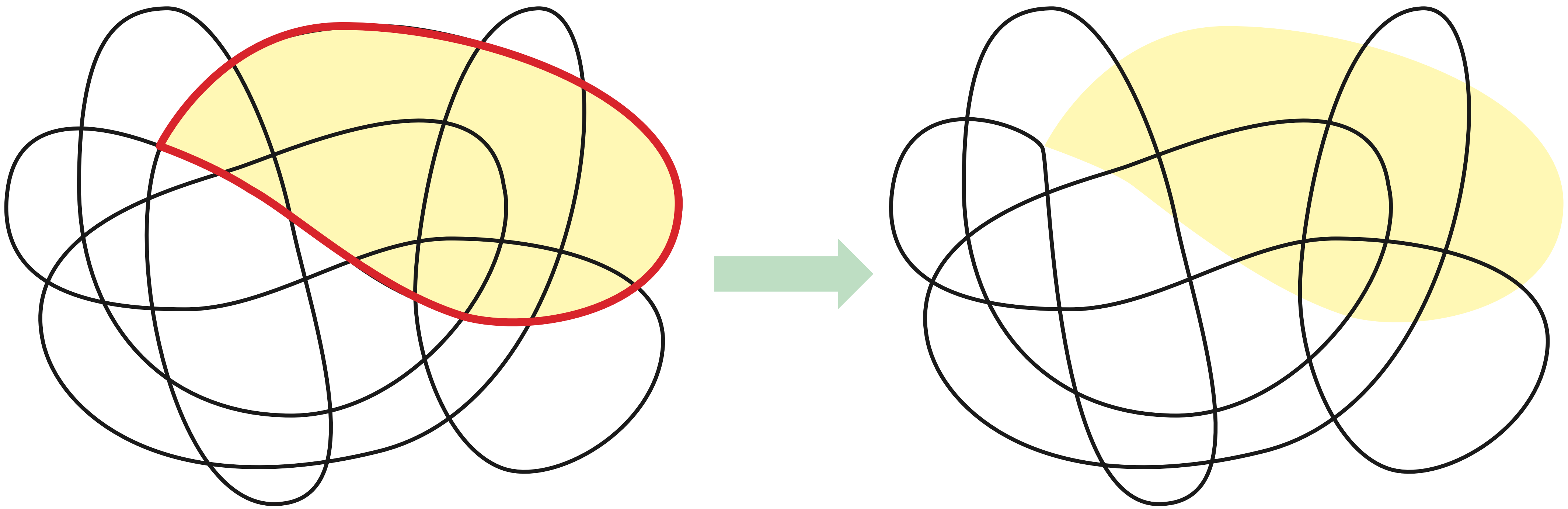
[Chang Erickson 2018]

- Every closed curve in the plane with n vertices can be simplified using $O(n^{3/2})$ homotopy moves, and this bound is tight in the worst case.
  - Upper bound via new algorithm
  - Lower bound via curve invariants

Planar Upper Bound

Shrinking loops

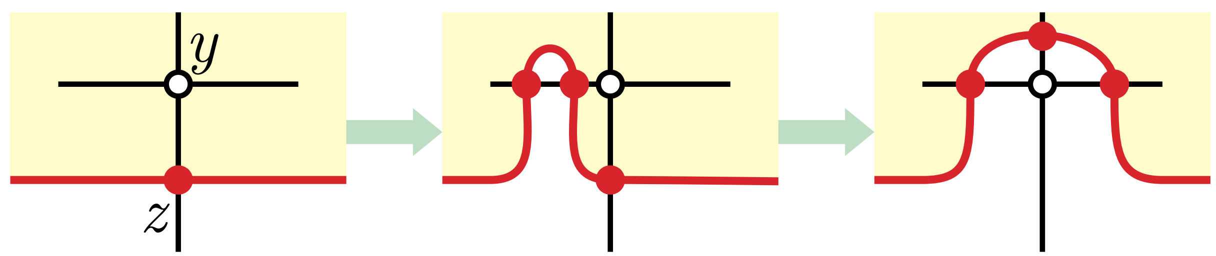
Any simple subloop can be removed using at most $3A$ homotopy moves, where A is the number of interior faces.



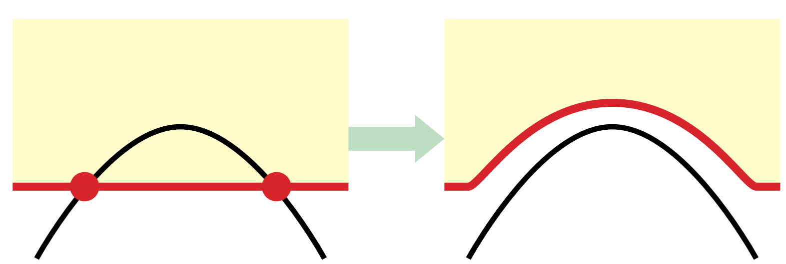
Shrinking loops

Any simple subloop can be removed using at most 3*A* homotopy moves, where *A* is the number of interior faces.

- If the loop contains any vertices, we can remove one vertex with one 0→2 move followed by one 3→3 move.



- If the loop bounds any empty bigons, we can remove one edge (and one empty bigon face) with one 2→0 move.



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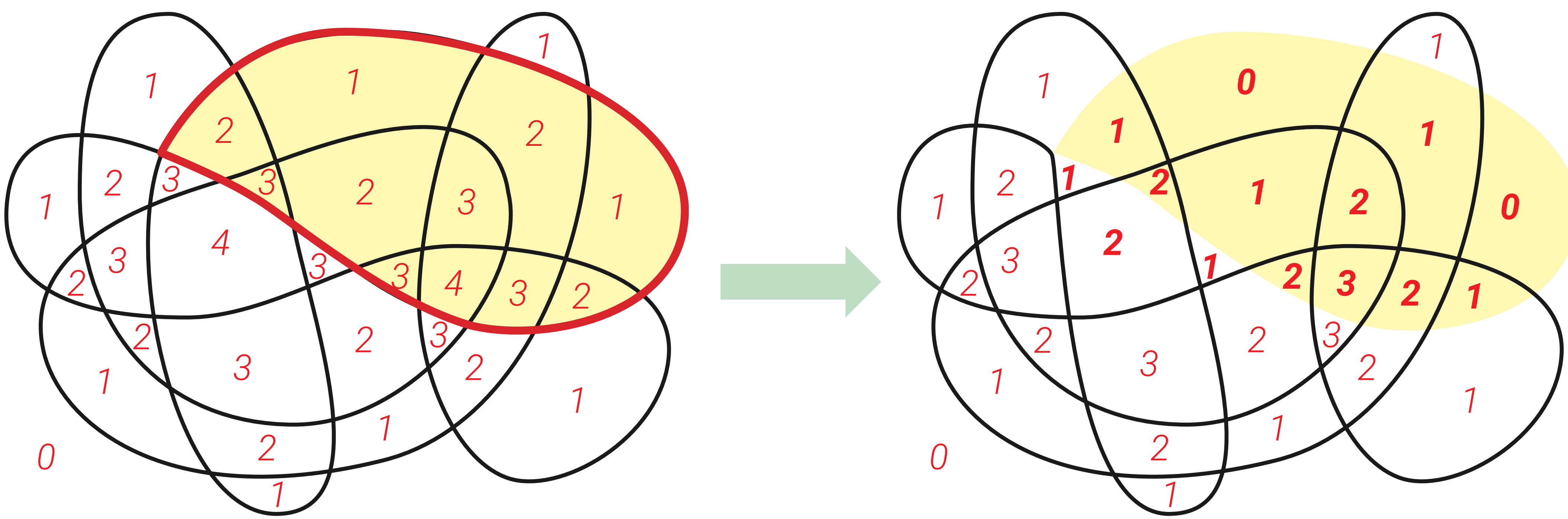
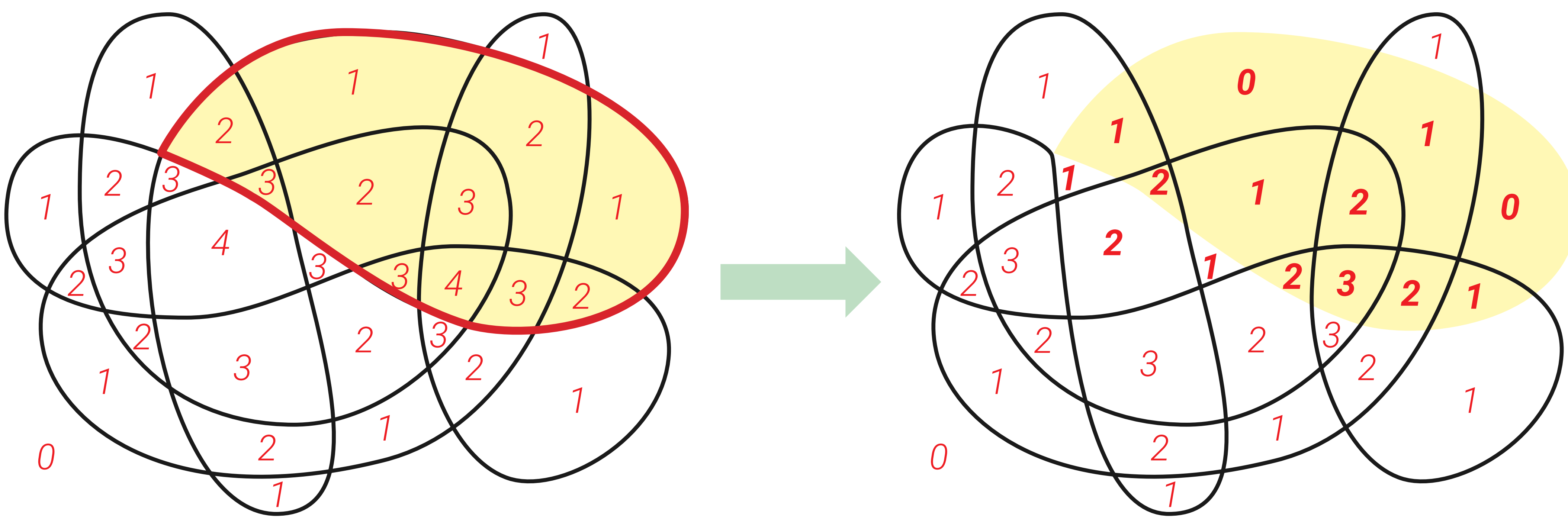
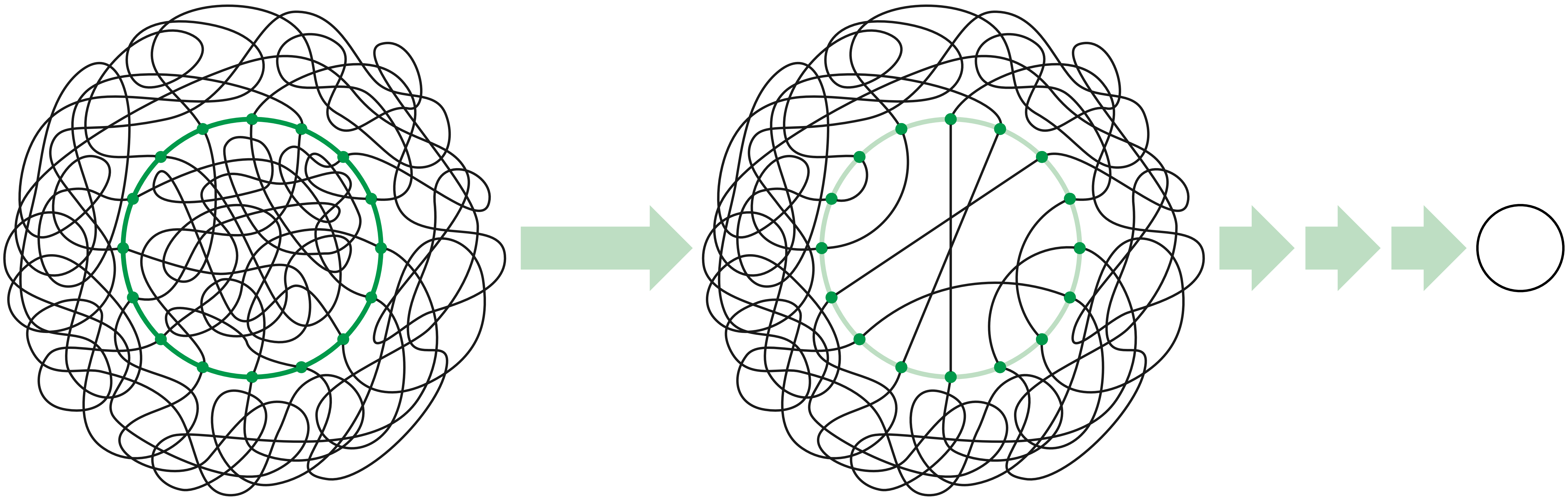


Contracting the loop decreases the sum of face depths by at least A.

New slow algorithm

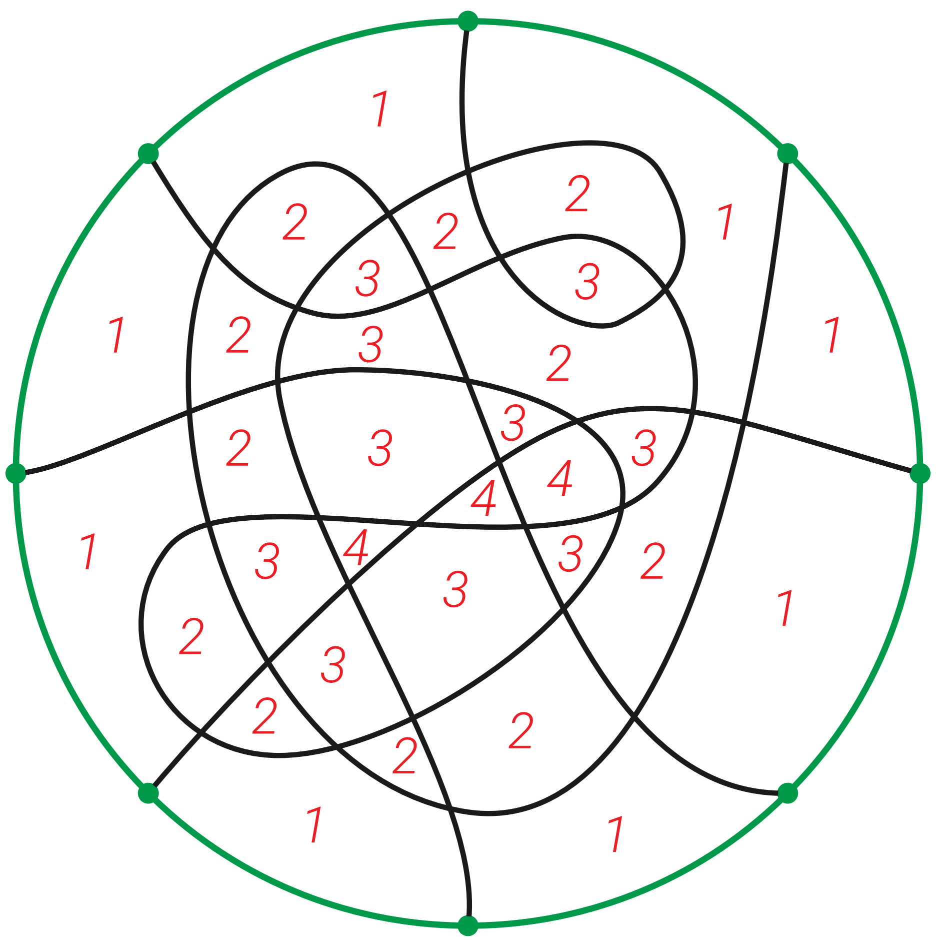
- While there are vertices, shrink any simple subloop.
- $O(\Phi) = O(n^2)$ moves, where Φ = potential





Tangle

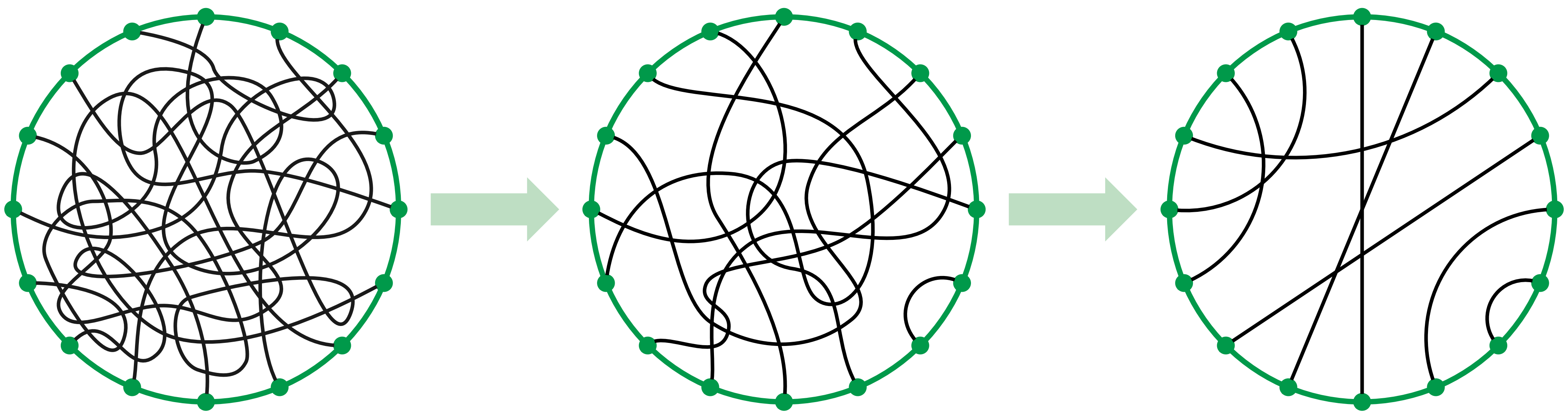
- Intersection of curve with a generic closed disk
- Boundary-to-boundary paths called ***strands***
- Face depths defined exactly as for curves.



Tightening tangles

Any tangle can be *tightened* in $O(md + ms)$ moves, where m = #vertices, d = max face depth, and s = #strands

- First remove all simple subloops in $O(\Phi) = O(md)$ moves.
- Then straighten all strands in $O(ms)$ moves.



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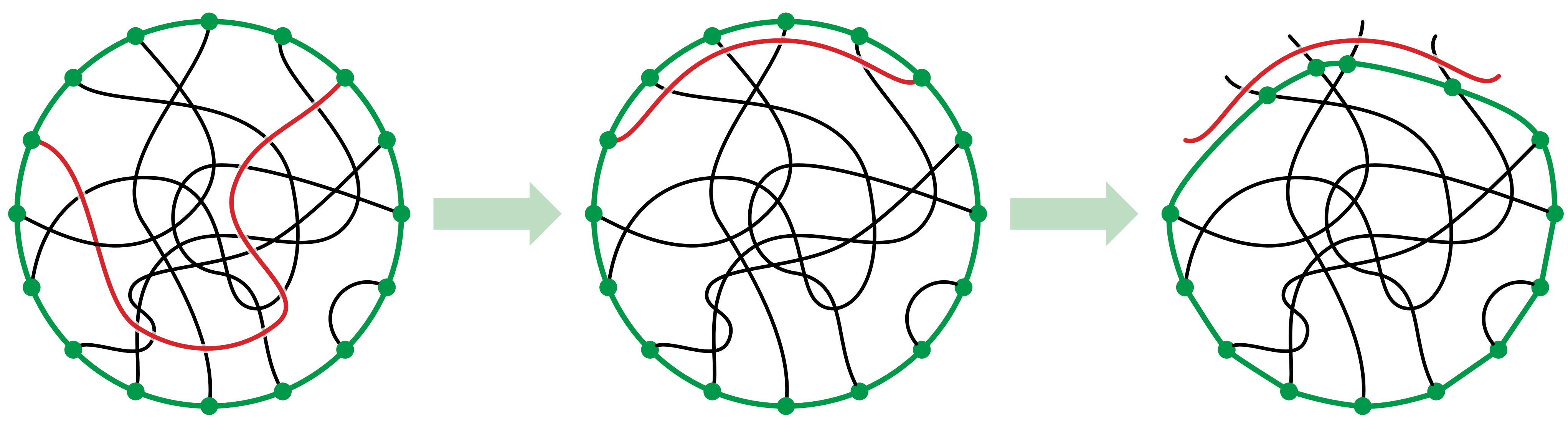
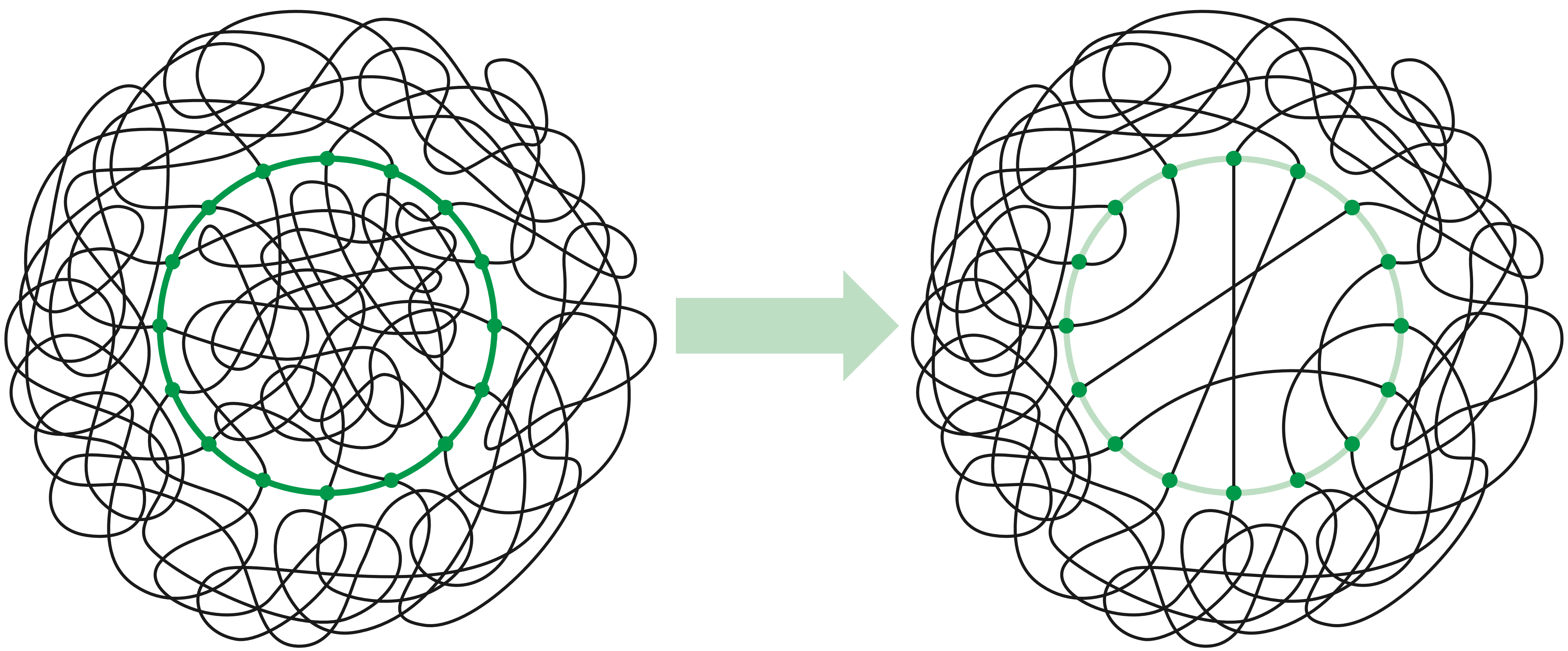


Useful tangles

We call a tangle *useful* if $m \geq s^2$ and $d = O(m^{1/2})$

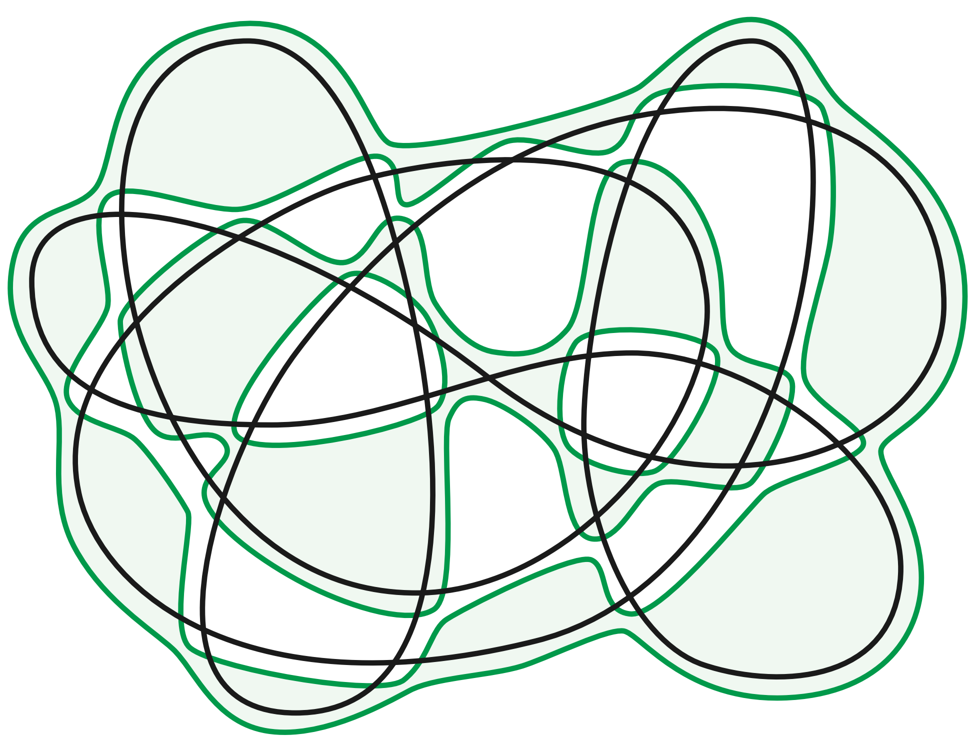
- Can be tightened using $O(md + ms) = O(m^{3/2})$ moves
- Tightening removes at least half of the interior vertices



Useful tangles

Lemma: Every curve admits a useful tangle.

- Depth contours define a sequence of nested tangles
- Suppose ith tangle has m_i vertices and s_i strands
- If the first i tangles are all useless, then $s_i \geq i/2$ and thus $m_i \geq i^2/4$



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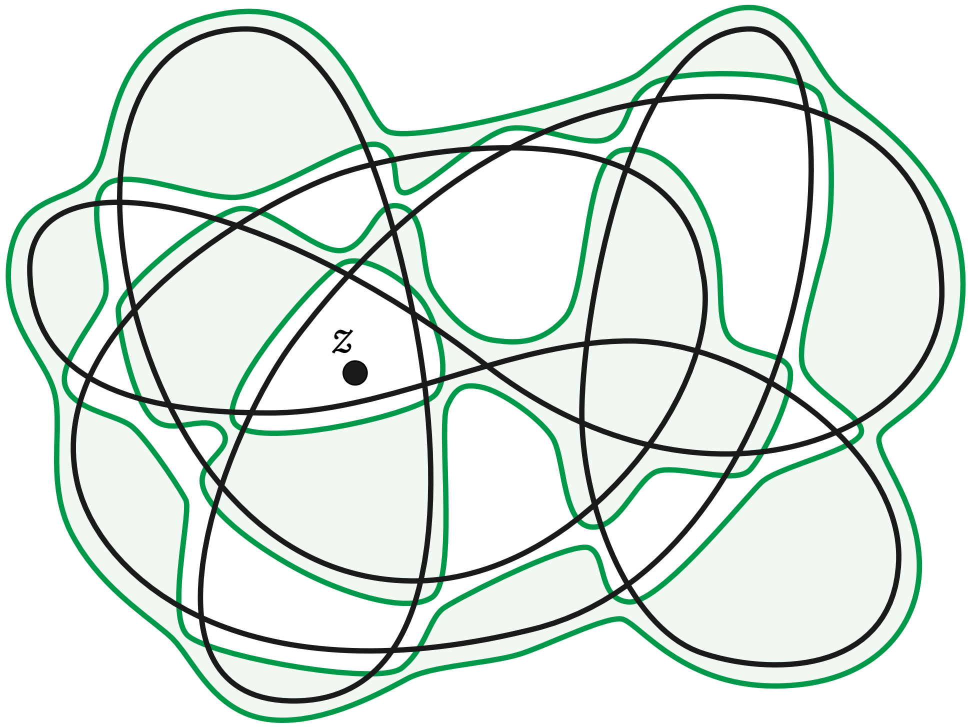


New faster algorithm

- While there are vertices, tighten any *useful* tangle
- Analysis:
  - Tightening a useful tangle with m vertices takes $O(m^{3/2})$ moves.
  - Charge $O(m^{1/2})=O(n^{1/2})$ moves to each deleted vertex
  - So removing all n vertices takes $O(n^{3/2})$ moves.

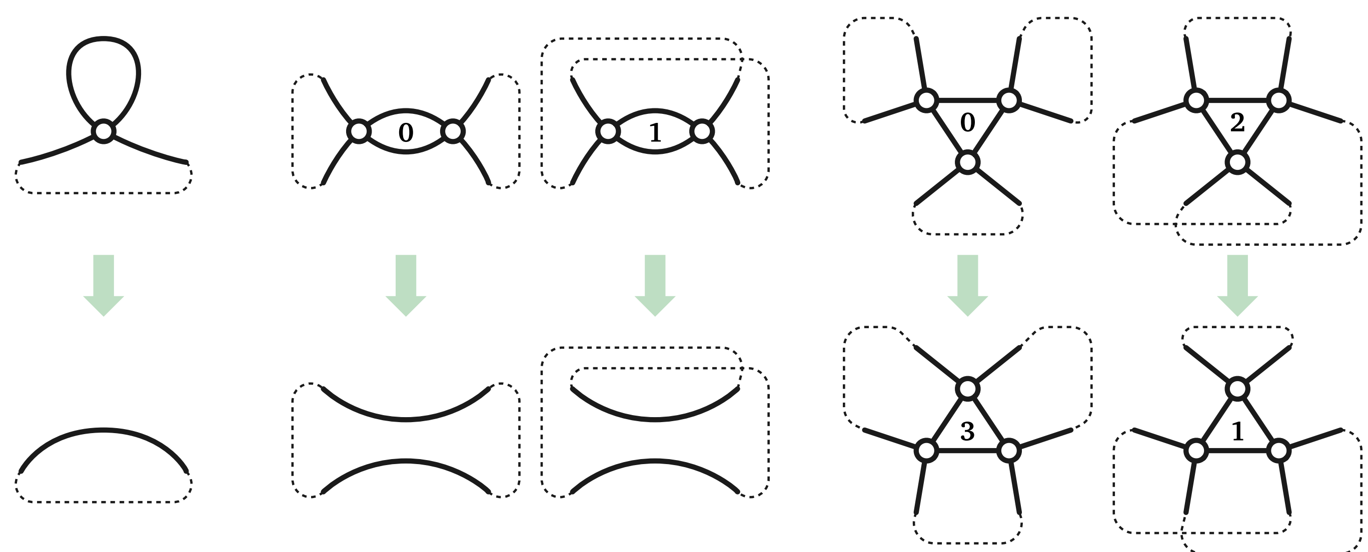
Planar Lower Bound

Defect

[Aicardi 1994] [Arnold 1994]

- Unique curve invariant that is zero for simple curves and changes as follows under homotopy moves:

1→0	2→0		3→3	



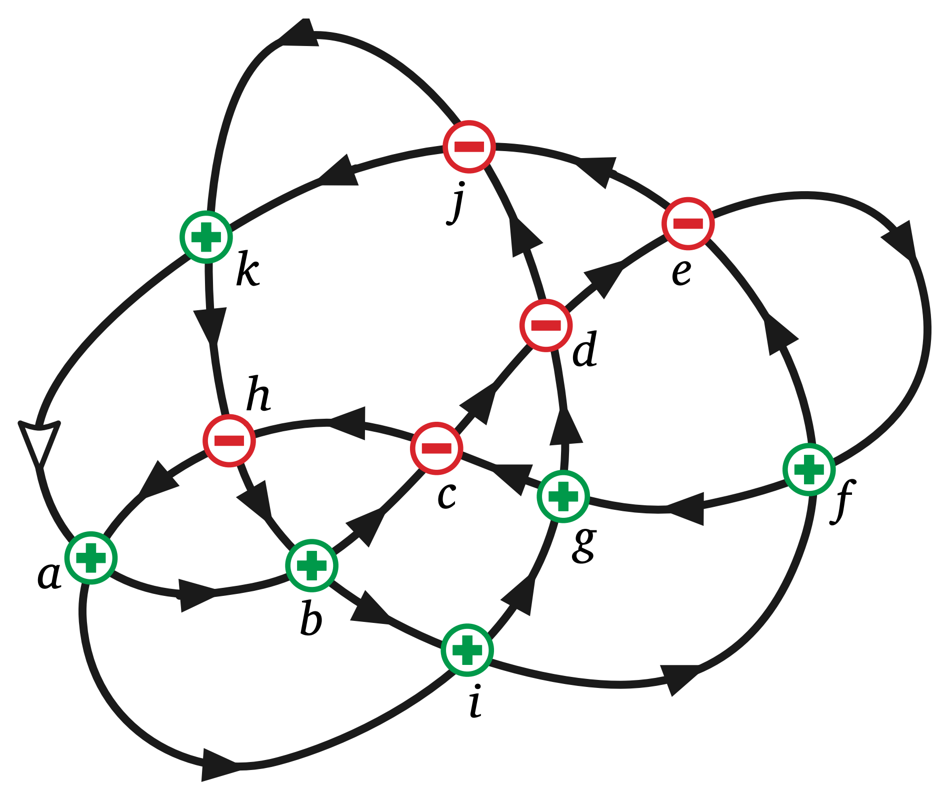
1→0	2→0		3→3	
0	0	−2	+2	+2

- Simplifying any curve γ requires at least |defect(γ)/2| homotopy moves.

Defect formula

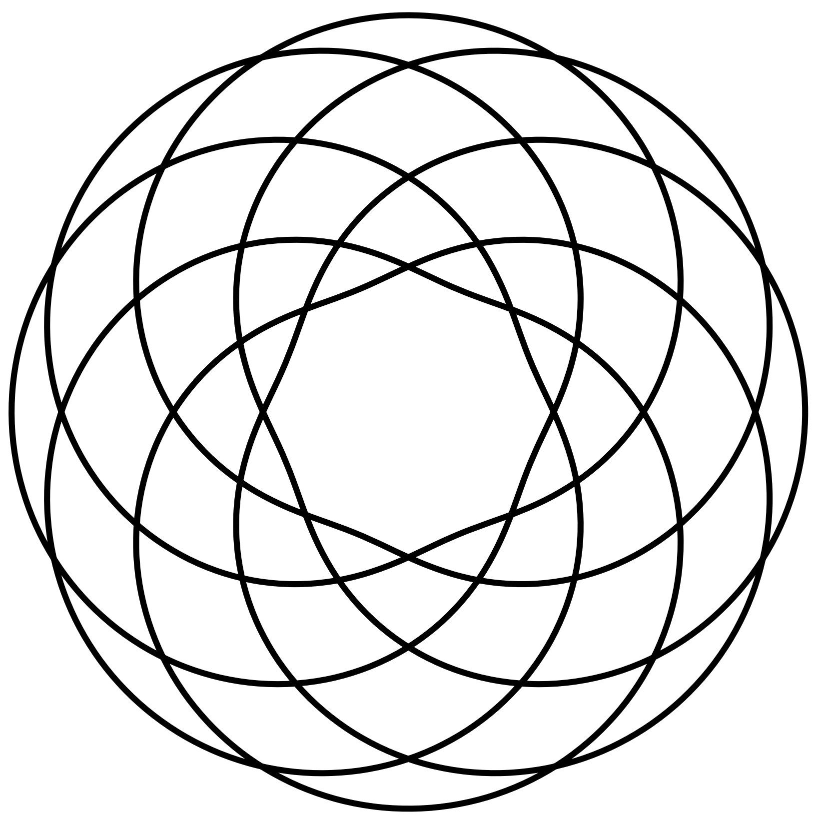
[Polyak 1998]

- defect = $-2 \sum_{x \between y} \operatorname{sgn}(x) \operatorname{sgn}(y)$
  - $x \between y$ means vertices x and y are ***interlaced***
  - $\operatorname{sgn}(x)$ is Gauss' sign convention for vertex x. *[Gauss c.1830]*
  - -2 is a historical artifact *[Arnold 1994]*

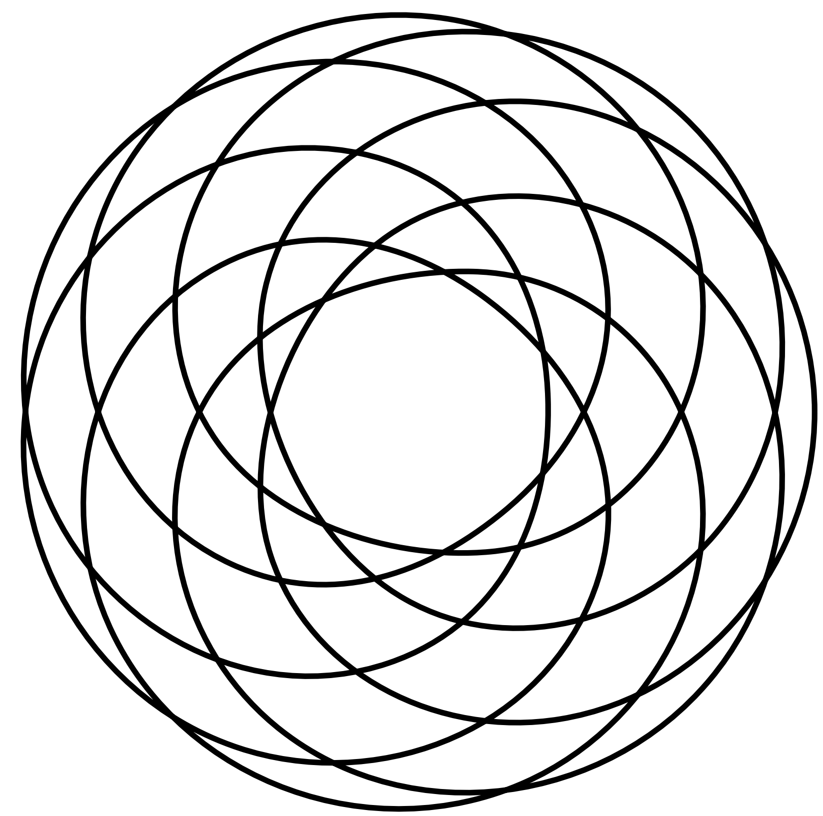


Flat torus knots

$$T(p,q)(\theta) := (\cos(q\theta)+2)\big(\cos(p\theta), \sin(p\theta)\big)$$



$T(7,8)$



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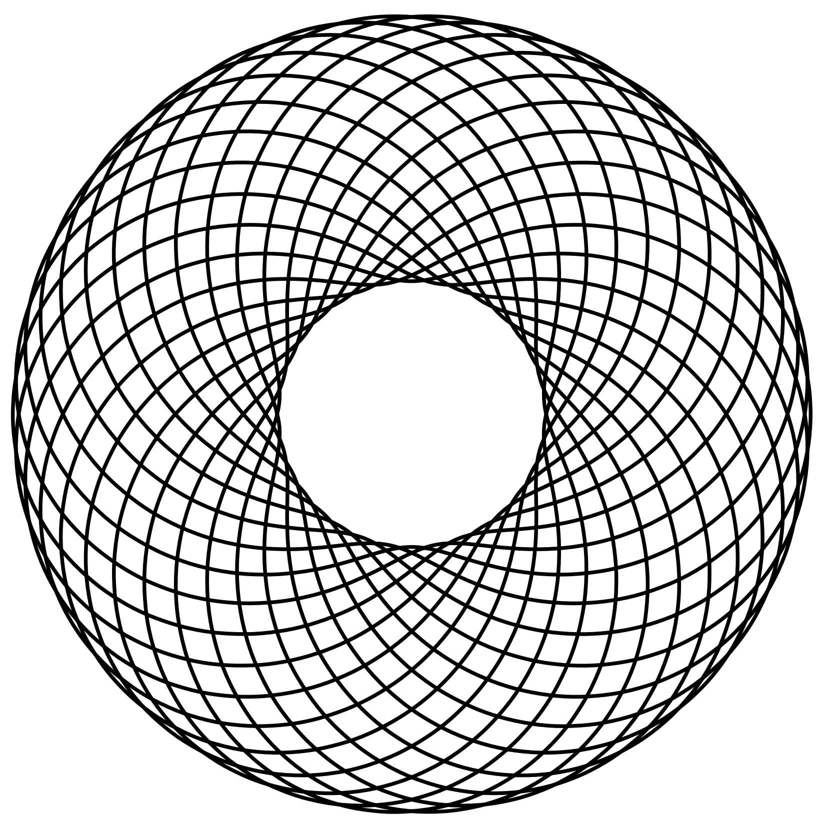
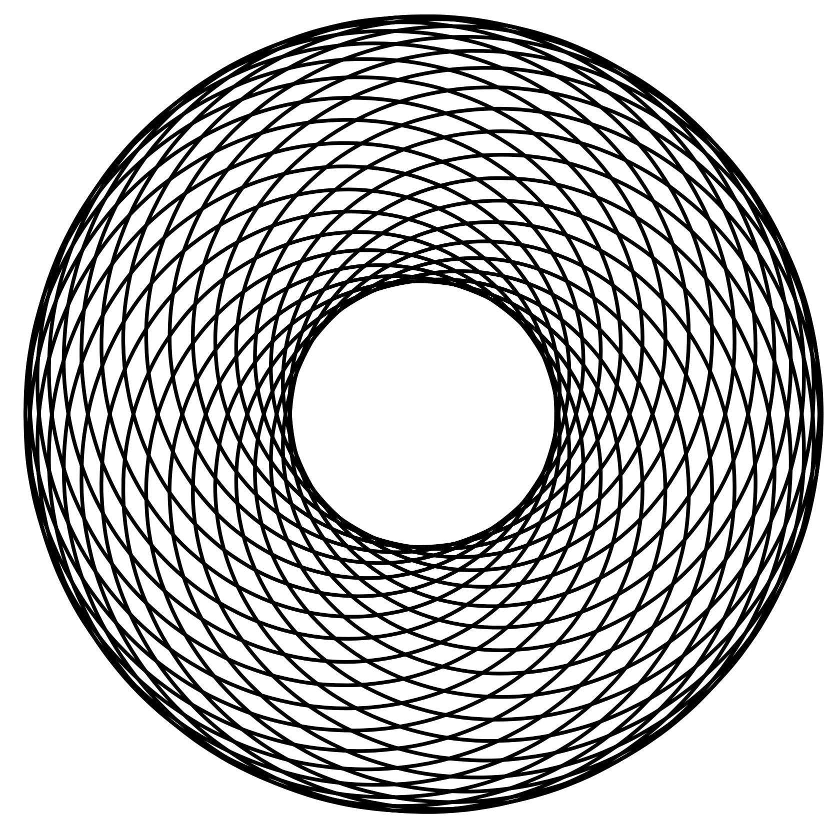


$T(21,34)$



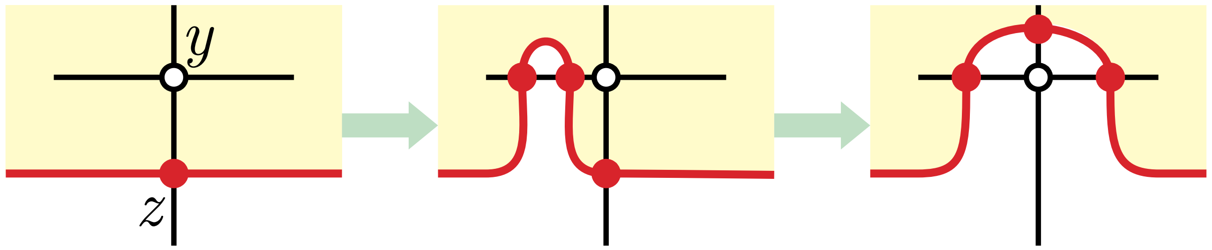
$T(34,21)$

Defects of flat torus knots

- $\operatorname{defect}(T(p, p+1)) = 2\binom{p+1}{3} = \Theta(n^{3/2})$ [Even-Zohar et al. 2014]
- $\operatorname{defect}(T(q+1, q)) = -2\binom{q}{3} = -\Theta(n^{3/2})$ [Hayashi et al. 2012]
- $\operatorname{defect}(T(p, ap+1)) = 2a\binom{p+1}{3} = \Theta(np)$
- $\operatorname{defect}(T(aq+1, q)) = -2a\binom{q}{3} = -\Theta(nq)$
- $\operatorname{defect}(T(F_{k+1}, F_k)) = \operatorname{defect}(T(F_{k-1}, F_k)) = (F_k - 1)(F_k - 2)/2 = \Theta(n)$

Open problem 1

- Previous O(n^2)-move algorithms are ***monotone***
  - They never perform 0→2 or 0→1 moves
  - So the number of vertices never increases.
- But our new O($n^{3/2}$)-move algorithm requires 0→2 moves



- Can every *n*-vertex closed curve be simplified using O($n^{3/2}$) ***non-increasing*** homotopy moves? Are Ω(n^2) moves sometimes necessary? Something in between?

Conjecture 1

- Every **loose tangle** with no empty monogons or bigons admits a 3→3 move that decreases the sum of face depths.
  - This would improve *Feo and Provan's* algorithm to $O(n^{3/2})$ moves!

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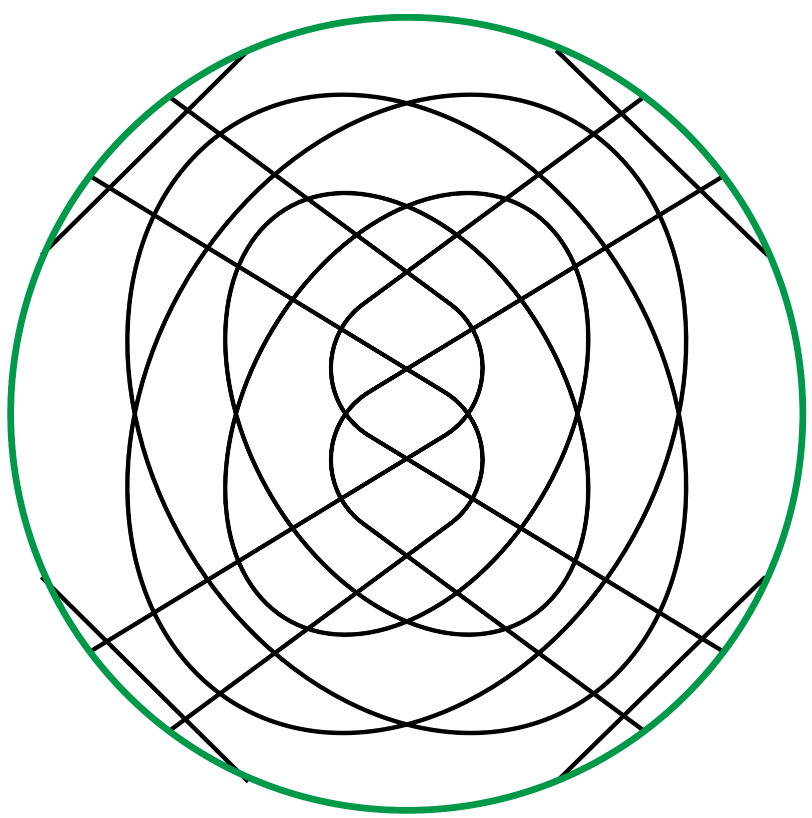
- Unfortunately, this conjecture is false!



Conjecture 2

- Every 4-regular plane graph has either an empty monogon or a bigon containing $O(n^{1/2})$ faces.
  - This would immediately improve *Steinitz's* algorithm to $O(n^{3/2})$ moves!

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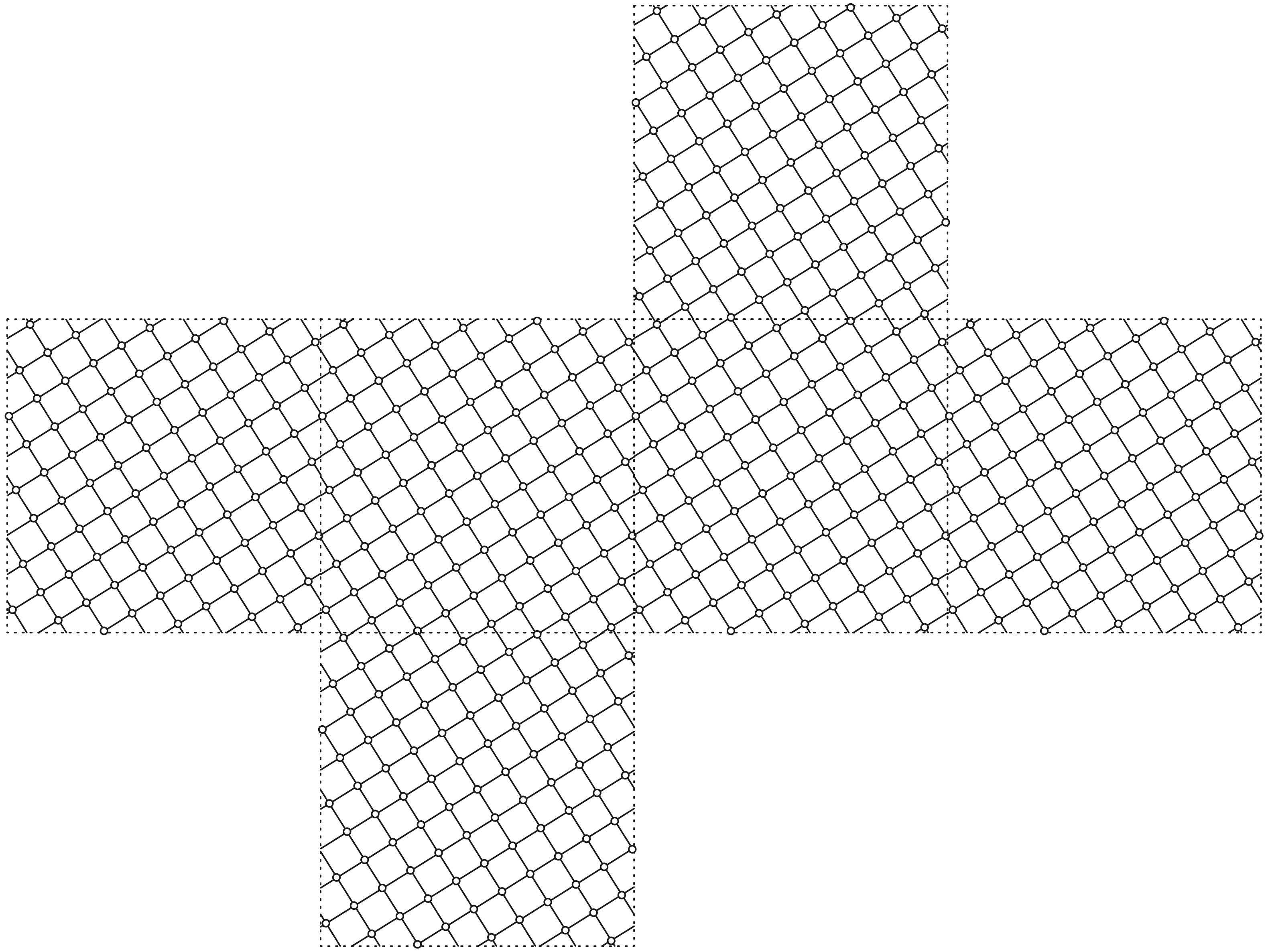
“Fibonacci cube”

Every loop and *every* bigon contains $\Omega(n)$ faces.



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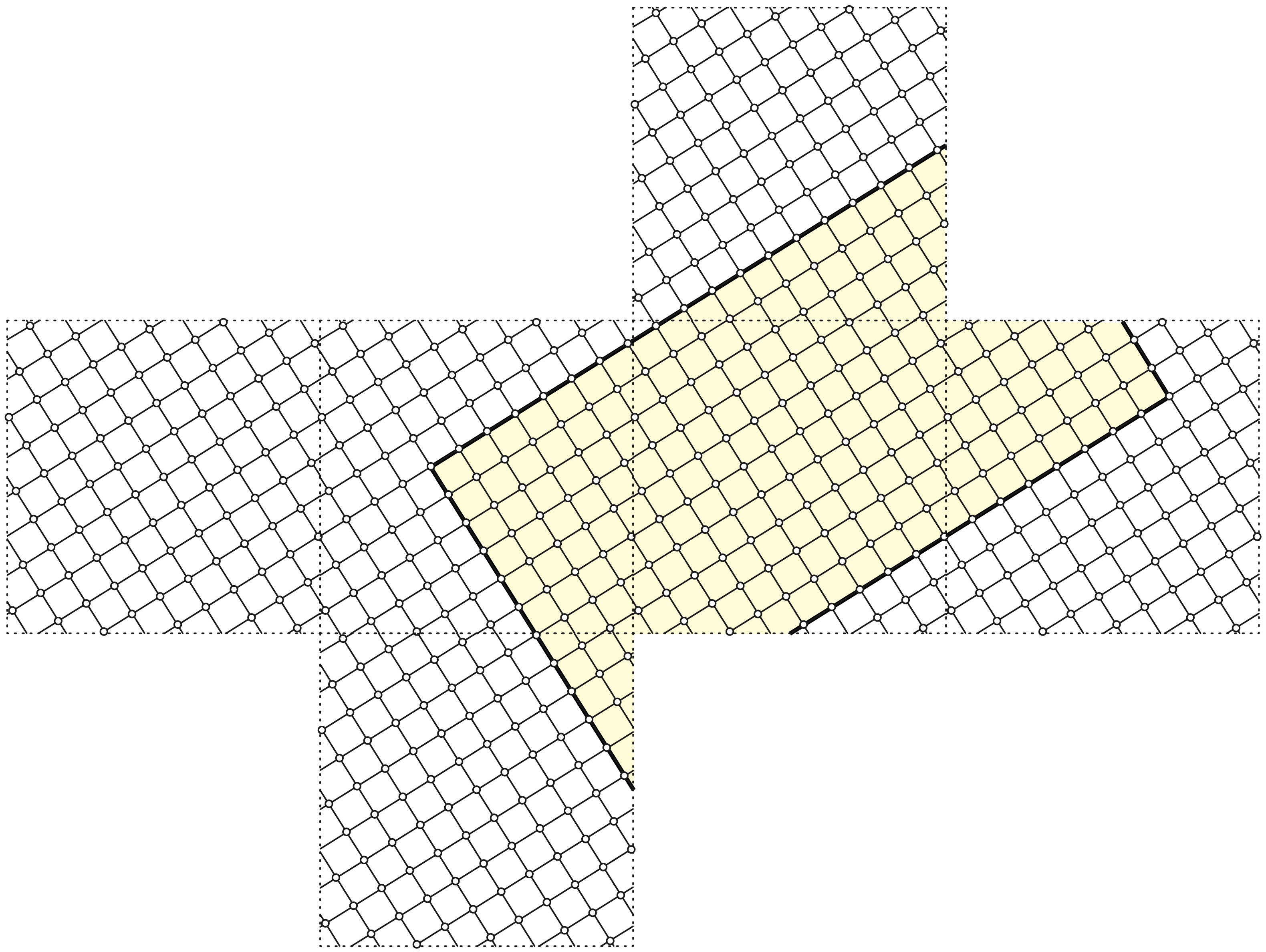

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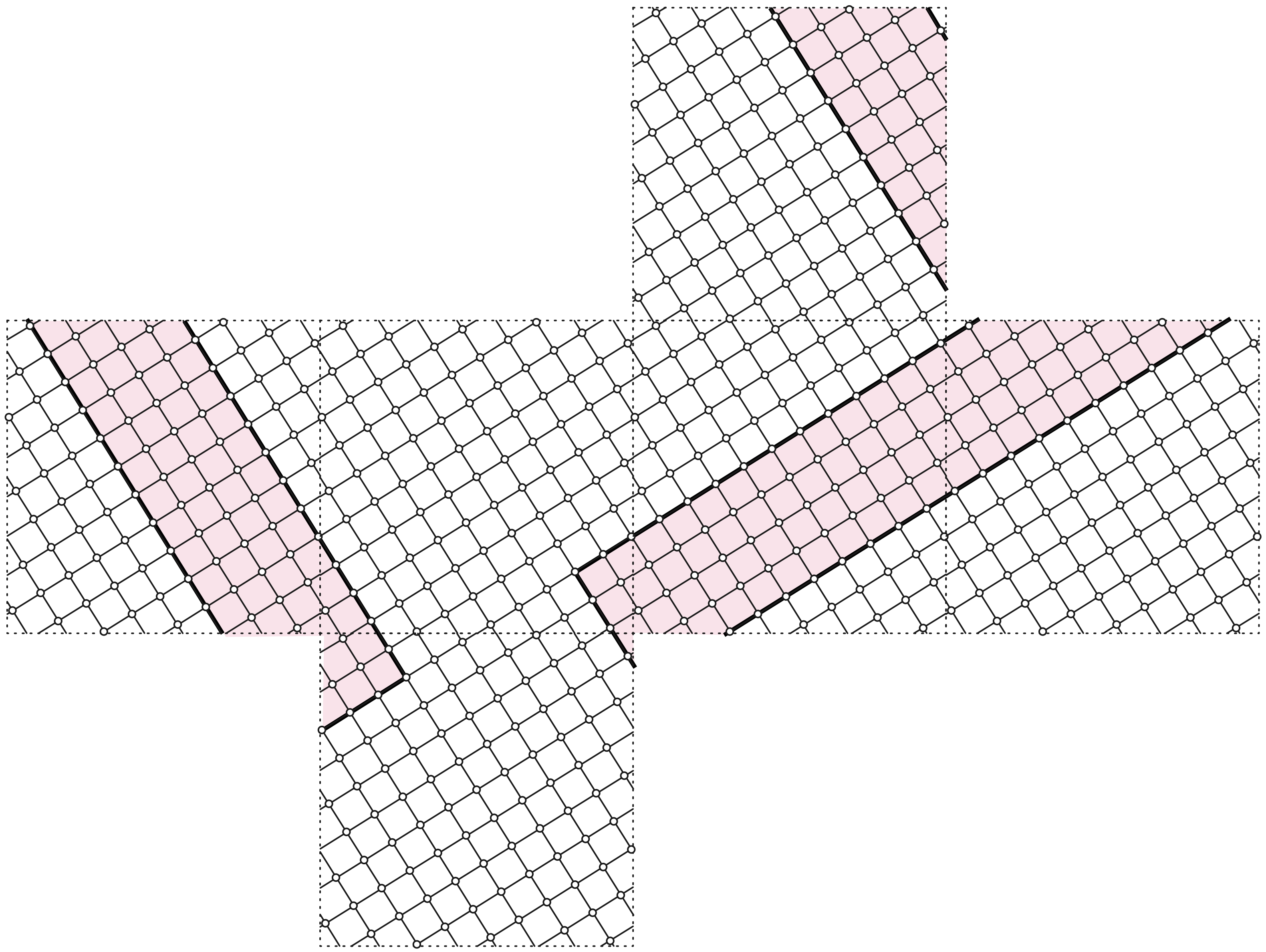
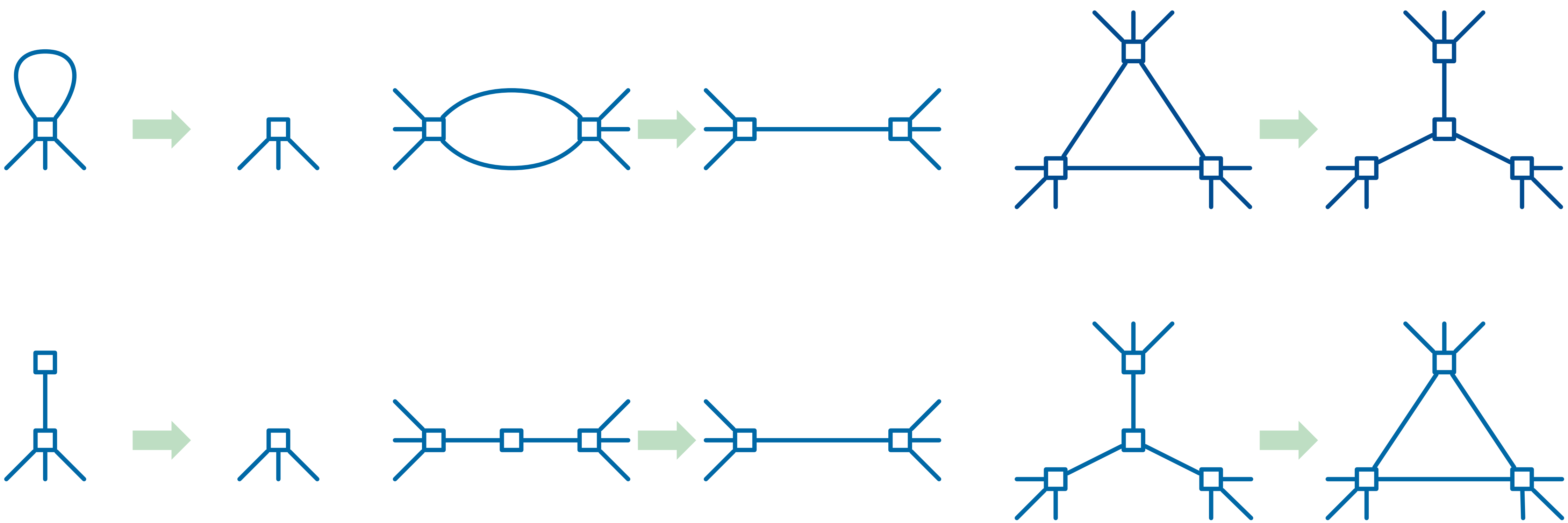


Electrical Reduction

Electrical transformations

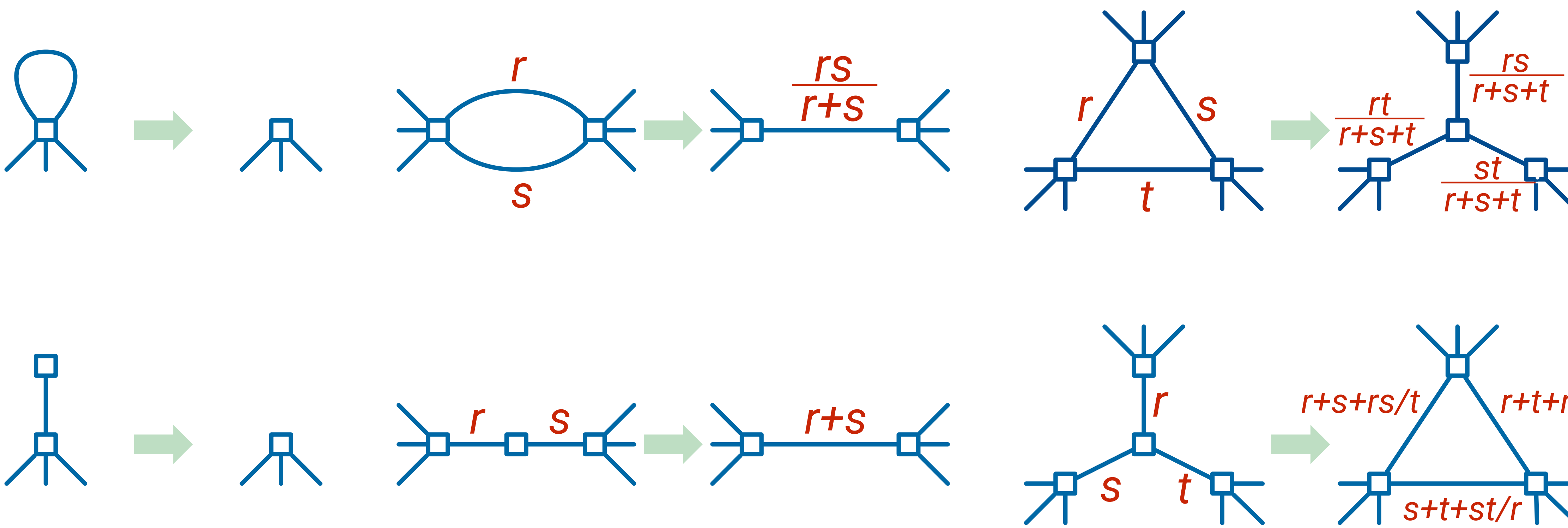
Loop reduction Parallel reduction Δ→Y transformation



Leaf reduction Series reduction Y→Δ transformation

Resistor network analysis

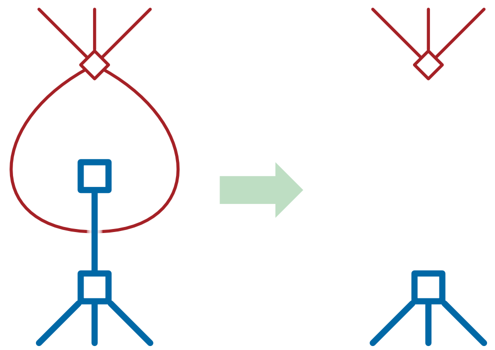
[Kennelly 1899]



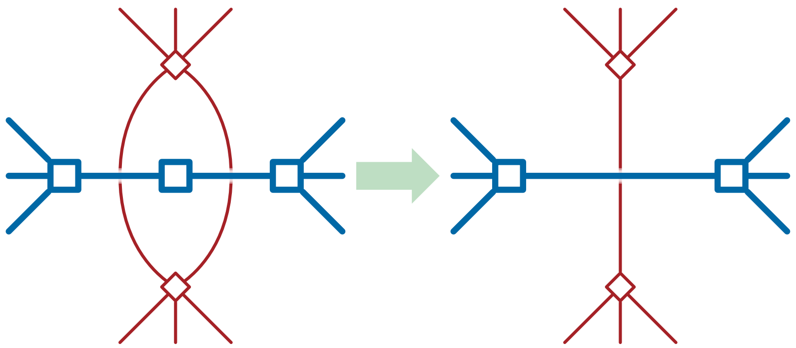
- Resistor network analysis *[Kennelly 1899]*
- AC circuit analysis *[Russell 1904]*
- Shortest paths and maximum flows *[Akers 60]*
- Network reliability estimation
  [Lehman 63, Traldi 83, Chari Feo Provan 96, Truemper 02]
- Multicommodity flows *[Feo 85]*
- Counting spanning trees, perfect matchings, and cuts
  [Colbourn Provan Vertigan 95]
- Generalized Laplacian linear systems
  [Gremban 96, Nakahara Takahashi 96]
- Circular planar networks
  [Colin de Verdière, Gitler, Vertigan 96; Curtis, Ingerman, Morrow 98]
- Kinematic analysis of articulated robots *[Staffelli Thomas 02]*
- Flow estimation from noisy measurements *[Zohar Geiger 07]*

Dual pairs

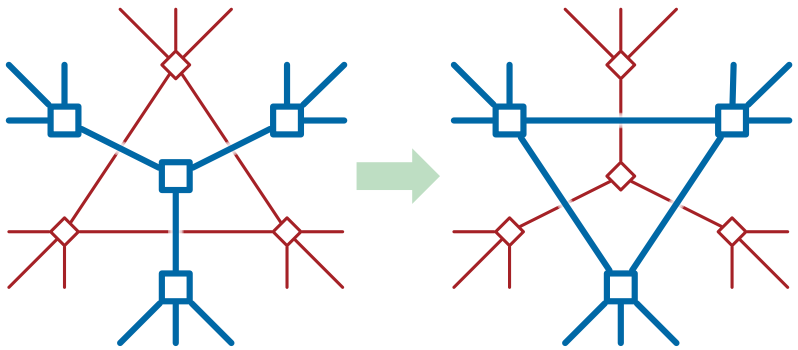
Degree-1 reduction



Series-parallel reduction



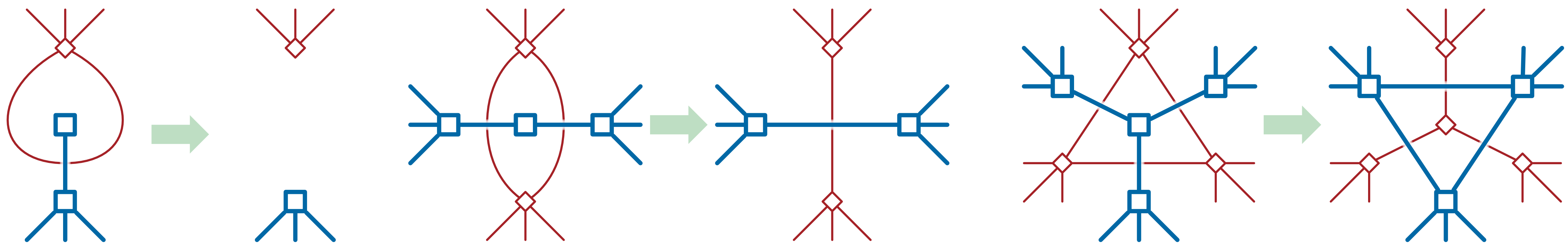
ΔY transformation



Electrical reduction

[Steinitz 1916, Epifanov 1966]

Any *planar* graph can be reduced to a single vertex using a *finite number* of electrical transformations.

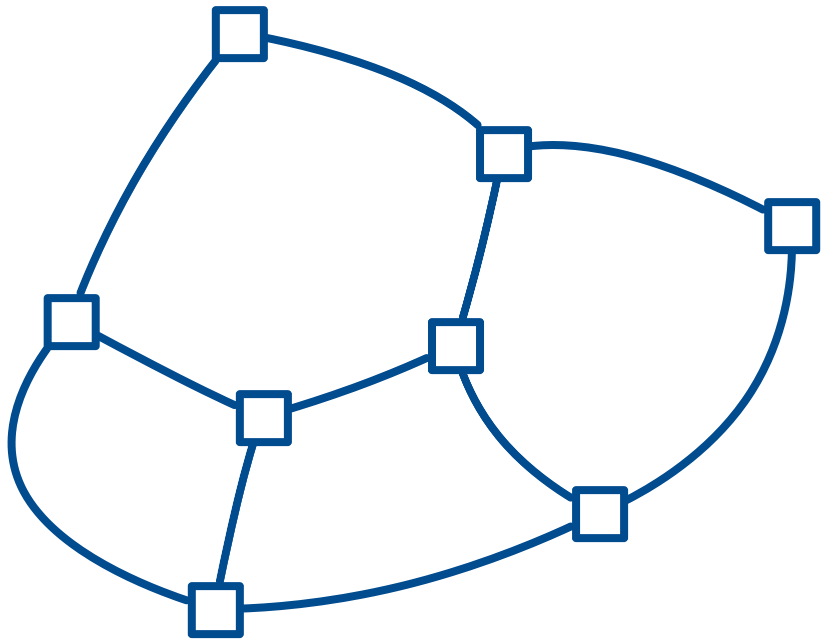


How many?

Medial graph $G^{\times}$

[Tait 1877, Steinitz 1916]

- Medial vertex for each edge of G
- Medial edge for each corner of G



Medial graph $G^\times$

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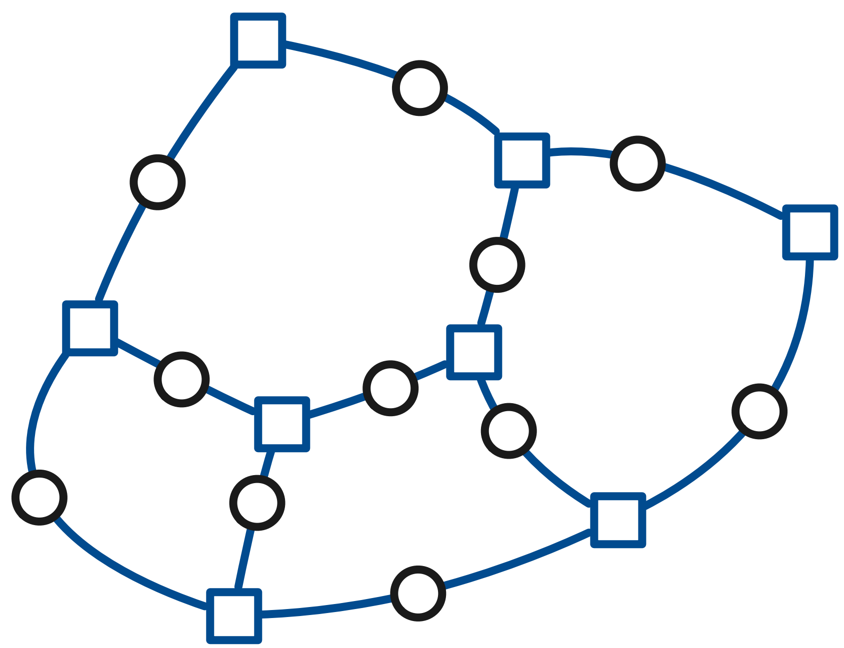

Medial graph $G^\times$

[Tait 1877, Steinitz 1916]

- Medial vertex for each edge of G
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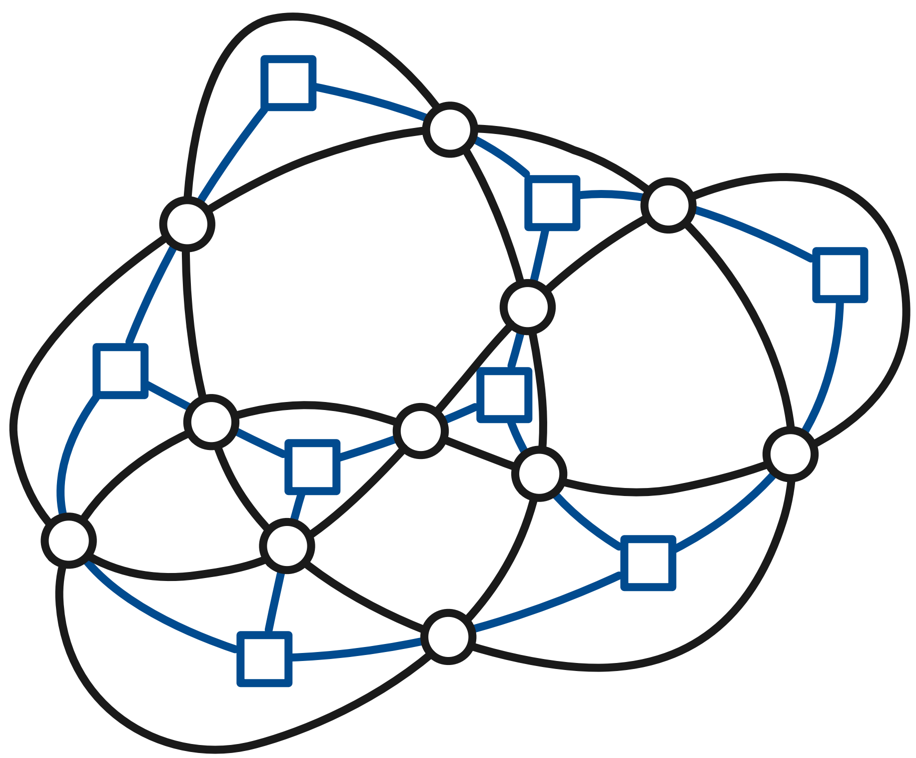

Medial graph $G^{\times}$

[Tait 1877, Steinitz 1916]

- Medial vertex for each edge of G
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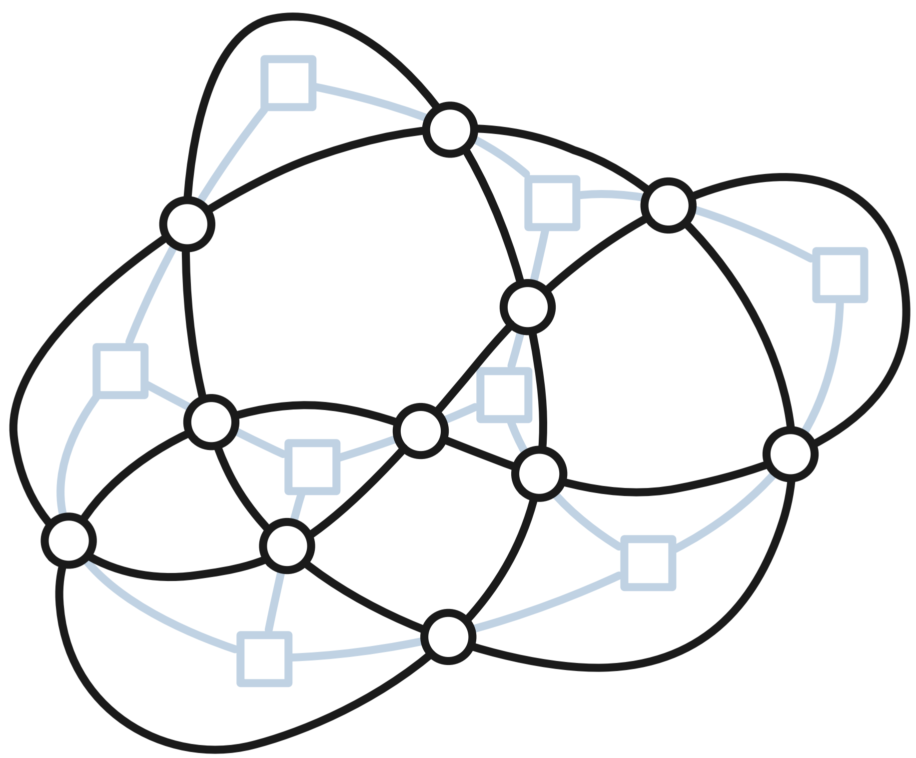
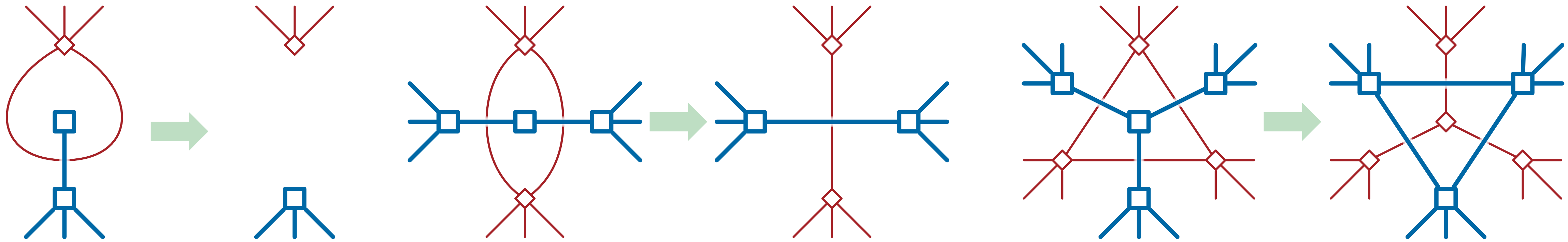


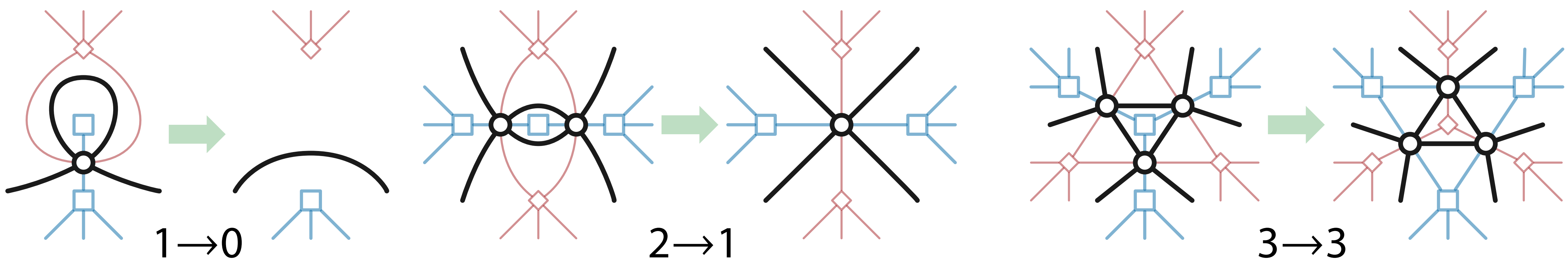
Medial electrical moves

[Steinitz 1916]



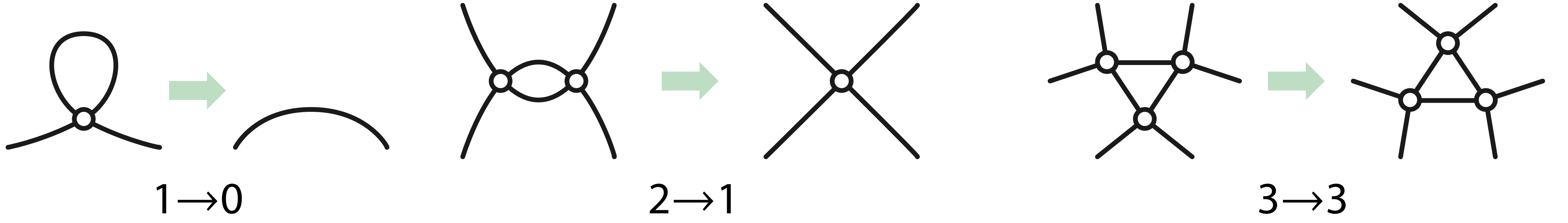
Medial electrical moves

[Steinitz 1916]



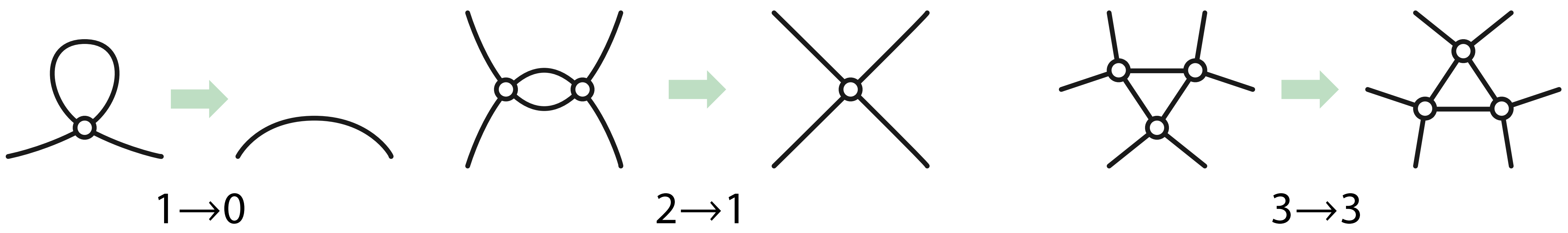
Medial electrical moves

[Steinitz 1916]

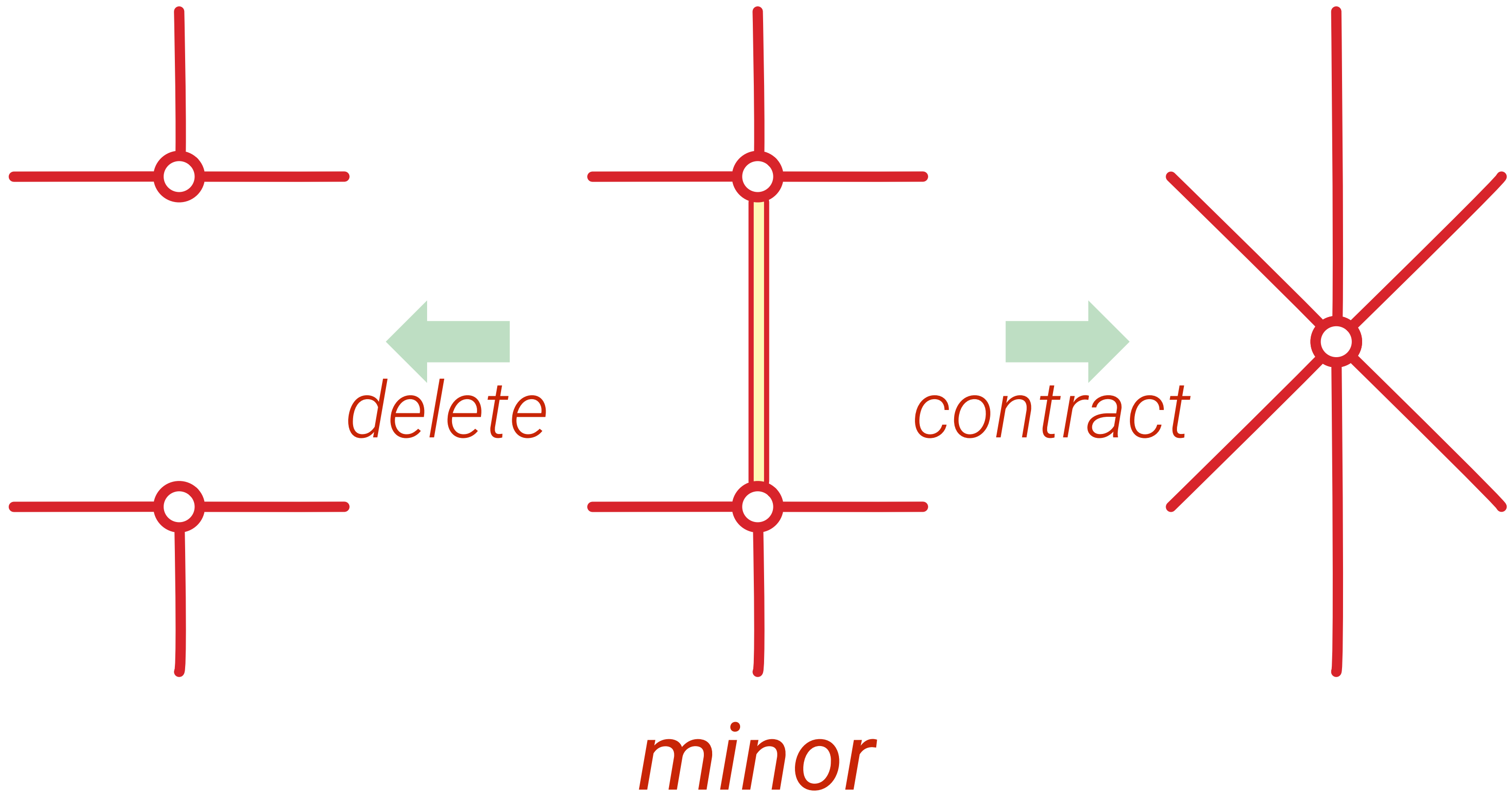


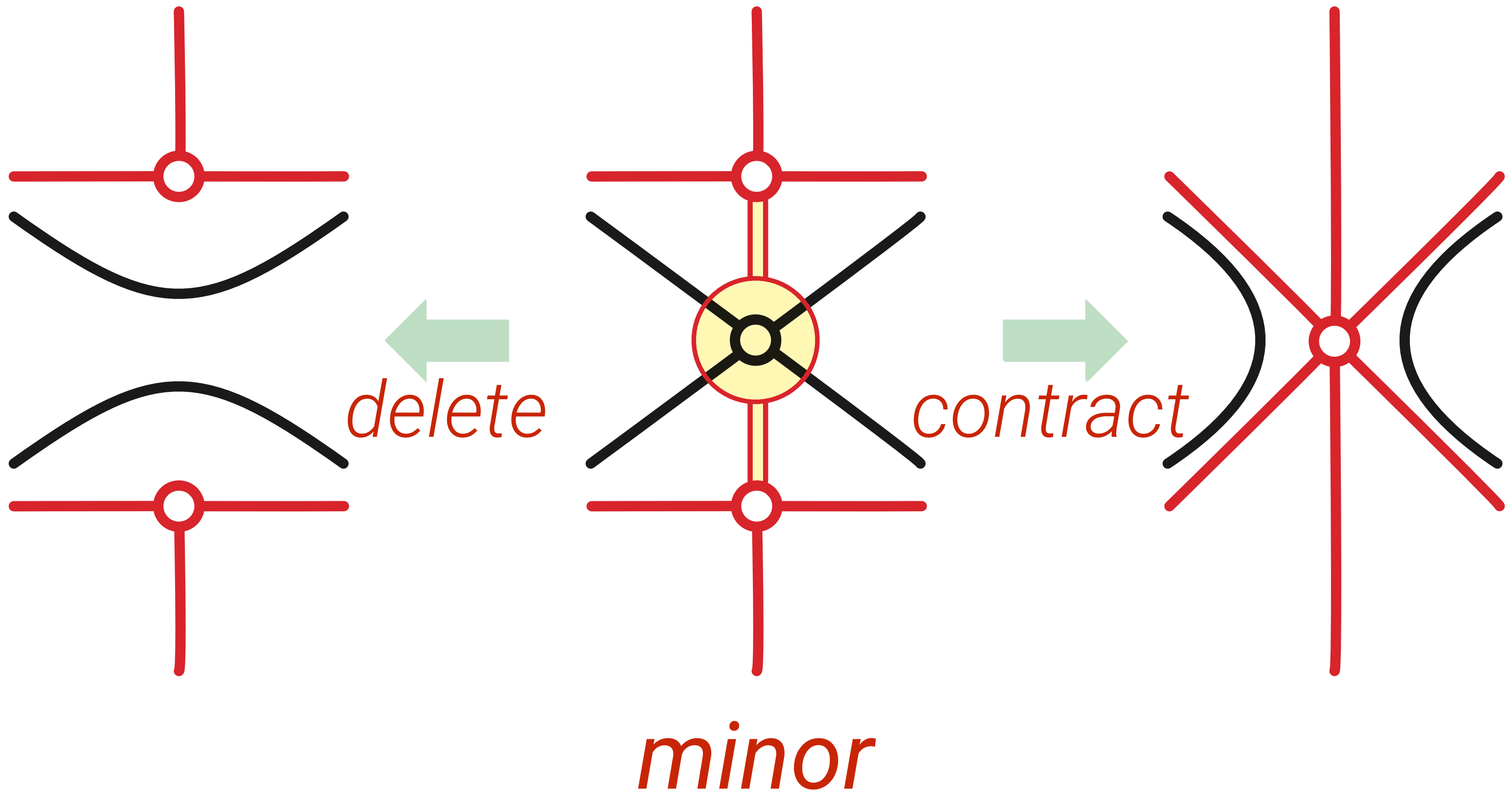
Medial electrical moves

[Steinitz 1916]



Hey, these look familiar.


delete
contract
minor


delete
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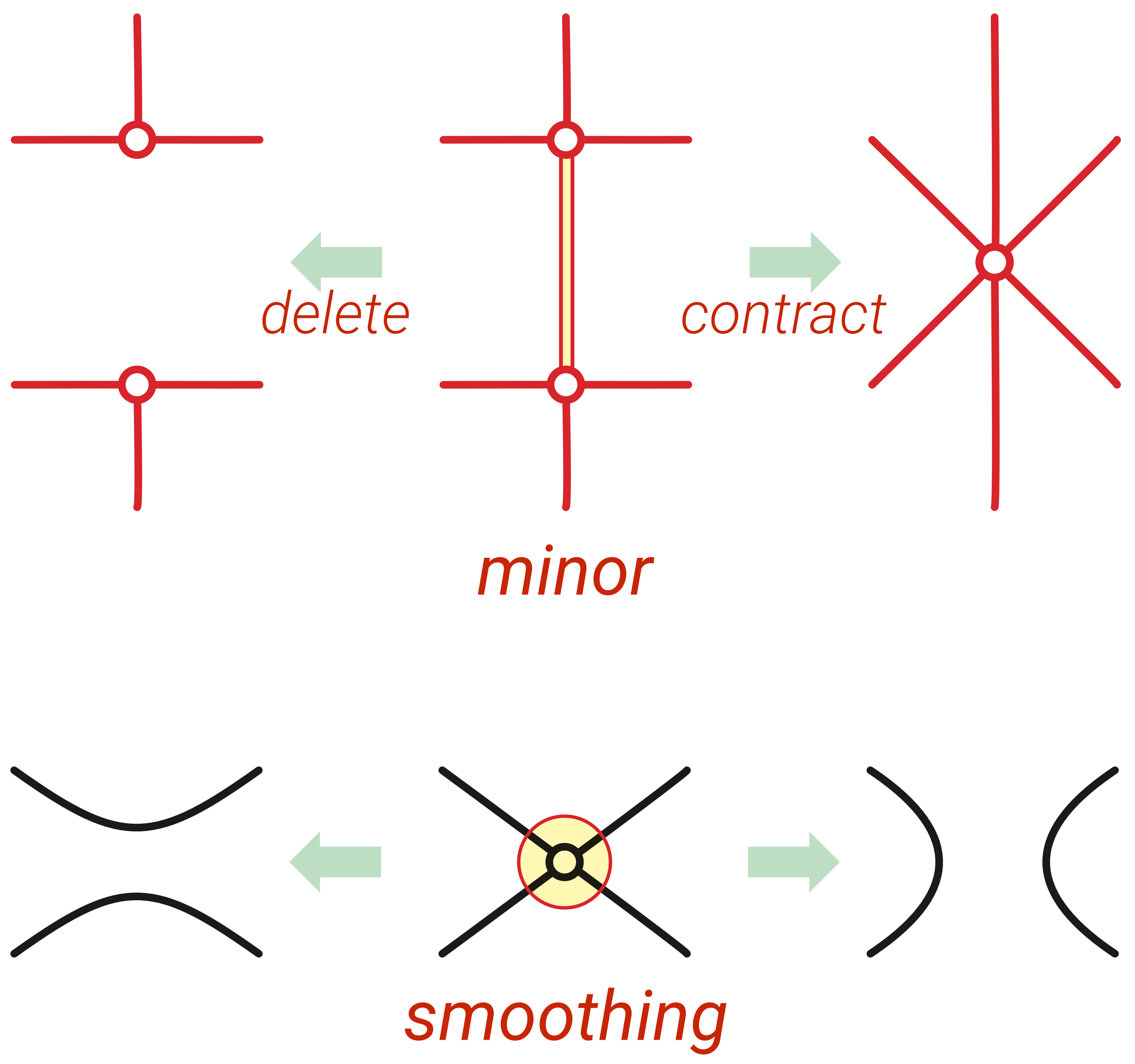
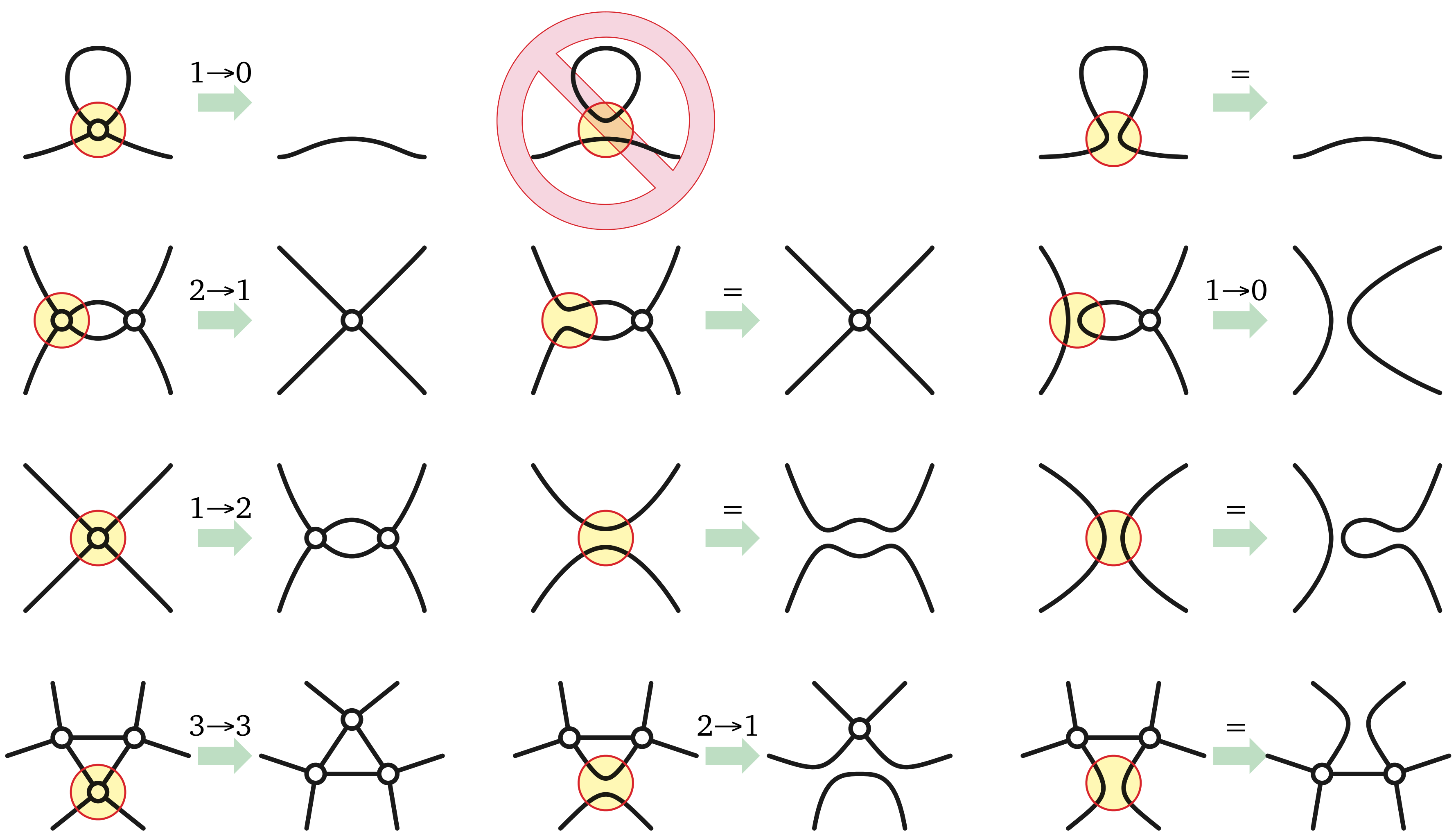

delete
contract
minor
smoothing

Smoothing Lemma

Reducing any *connected proper minor* of G requires *strictly fewer* electrical moves than reducing G.



[Truemper 1989, Gitler 1991]

Reduction Lemma

The minimum # electrical moves to reduce G is *at least* the minimum # homotopy moves to simplify $G^\times$.

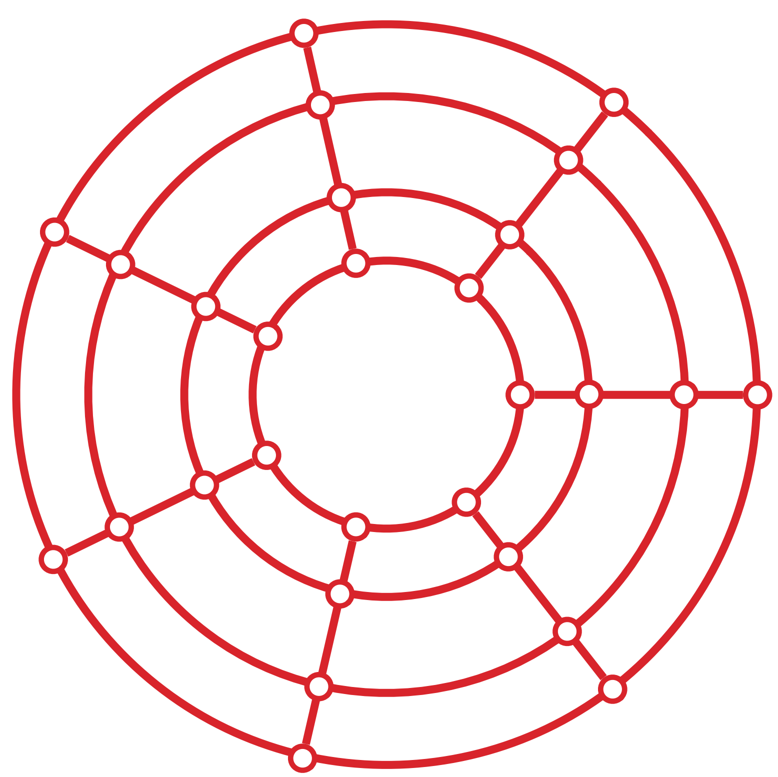
- **Proof:** Replace the first 2→1 move with a 2→0 move, then apply the minor lemma and induction.

[Noble Welsh 2000]

Electrical Lower Bound

[Chang Erickson 2018]

For all k, the $k \times (2k-1)$ cylindrical grid graph requires $\Omega(k^3) = \Omega(n^{3/2})$ electrical moves to reduce.

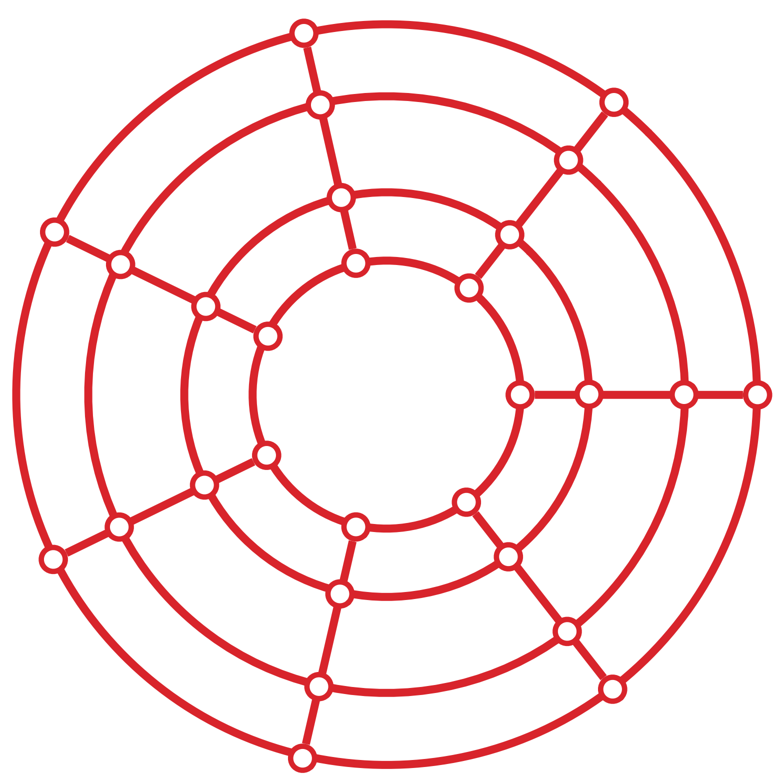


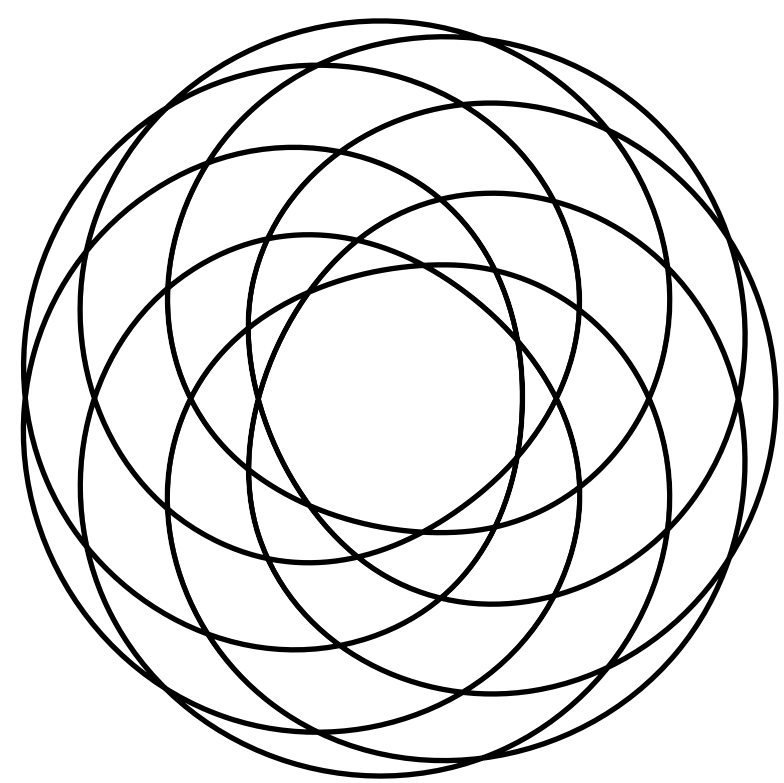
Electrical Lower Bound

[Chang Erickson 2018]

For all k, the $k \times (2k-1)$ cylindrical grid graph requires $\Omega(k^3) = \Omega(n^{3/2})$ electrical moves to reduce.

- Proof: Its medial graph is $T(2k, 2k-1)$.





Higher-Genus Lower Bound

Lower bound

[Chang Erickson Letscher deMesmay Schleimer Sedgwick Thurston Tillmann 2018]

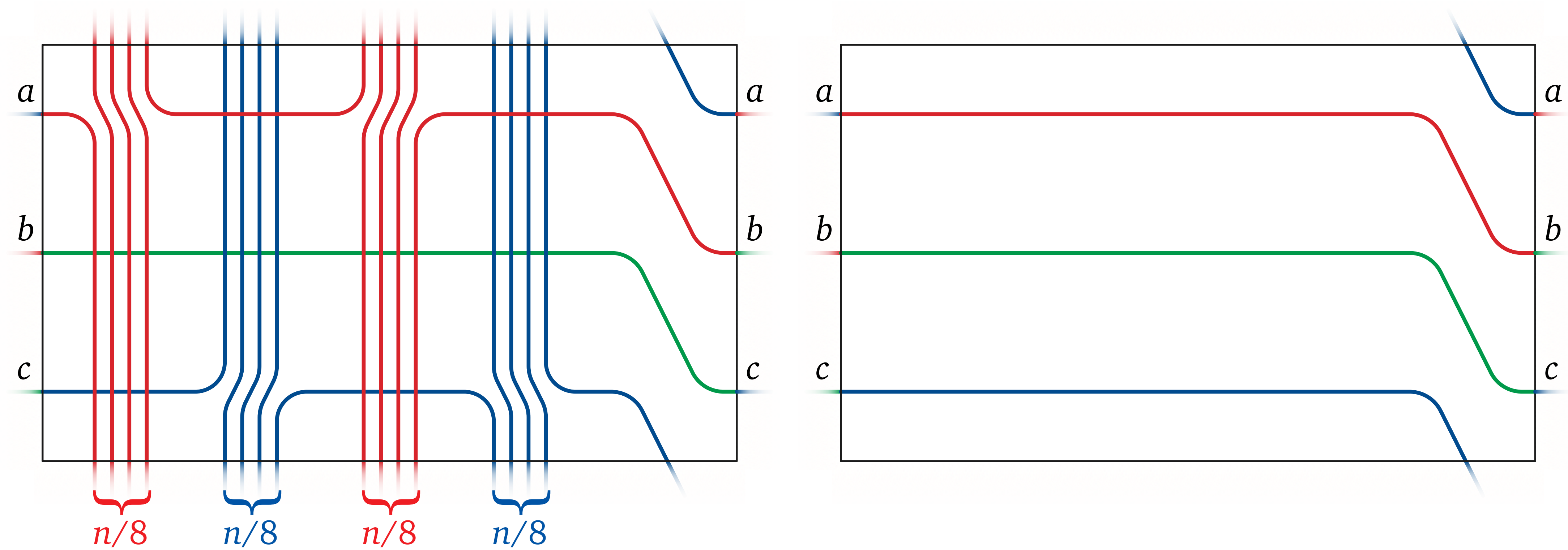
- On any surface with genus>0, simplifying a *contractible* curve requires $\Omega(n^2)$ homotopy moves in the worst case.

- Matches known $O(n^2)$ upper bound. *[Hass Scott 1994][Steinitz 1917]*
  - Either the curve is already simple or it has a loop or a bigon.

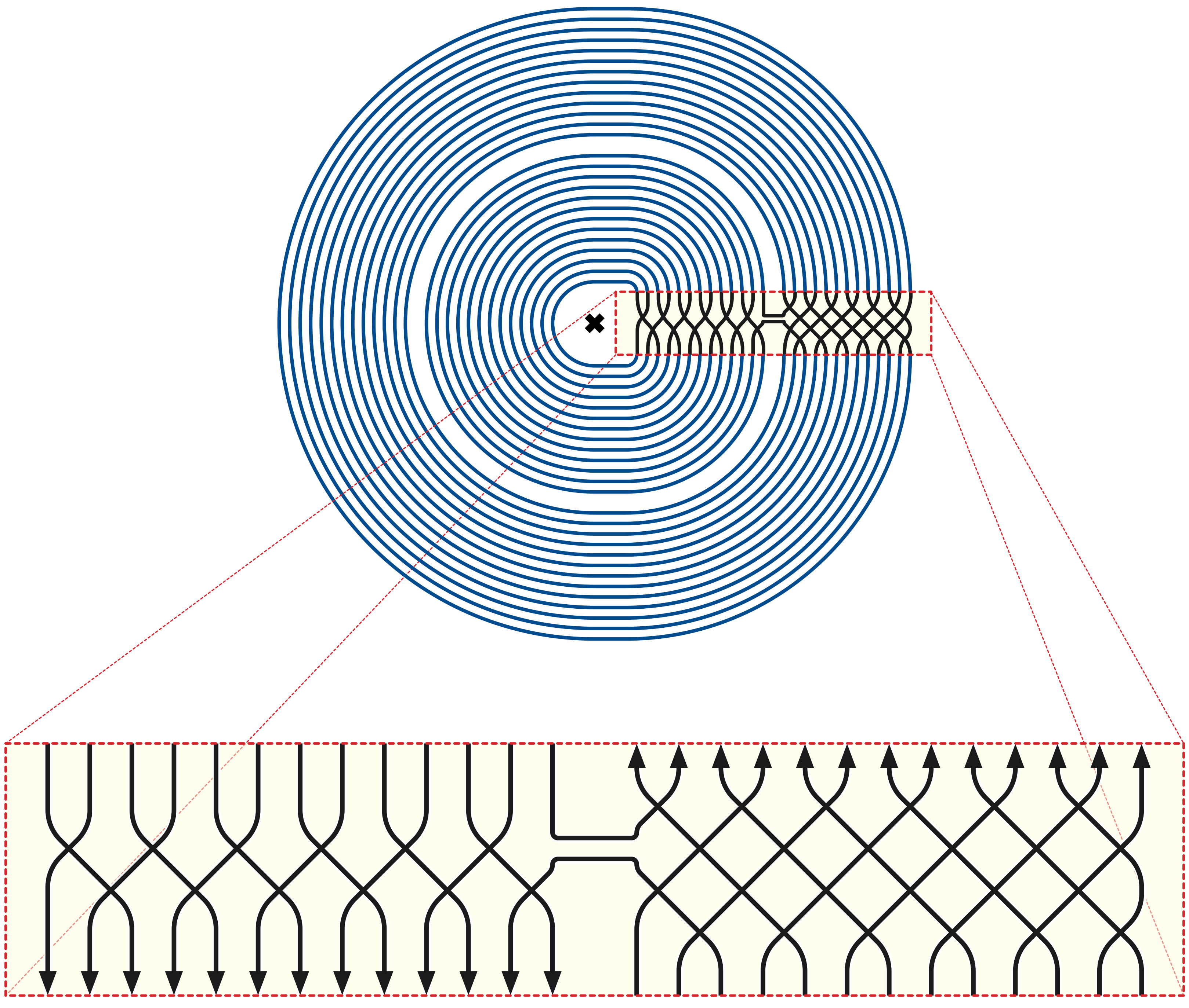
- It suffices to consider the *punctured plane* $\Sigma := \mathbf{R}^2 \backslash \{o\}$, where o is an arbitrary point called the *origin*.

Defect?

- There are curves on the torus with ***defect Ω(n²)***
  - ...but the examples we know are not contractible.

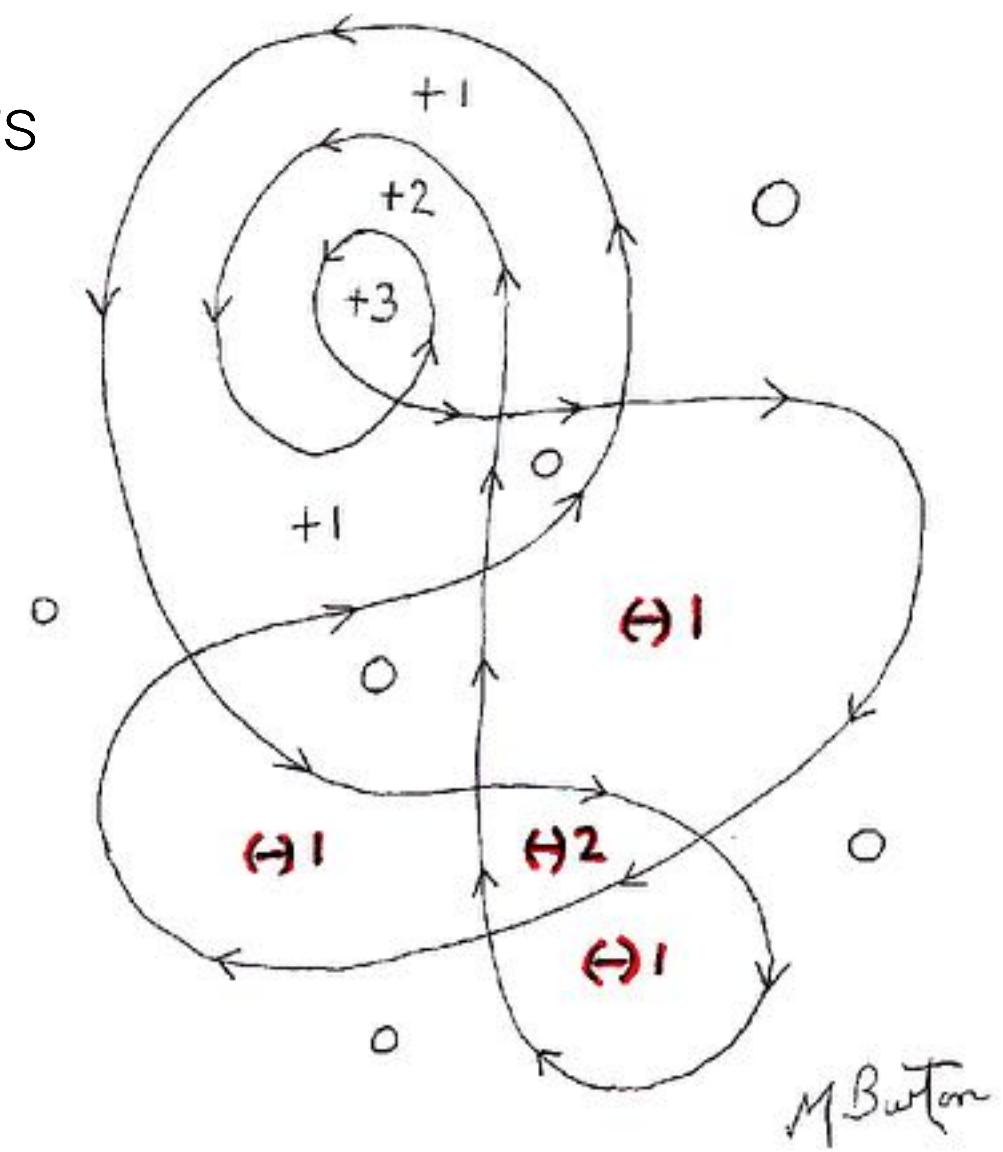


The bad curve



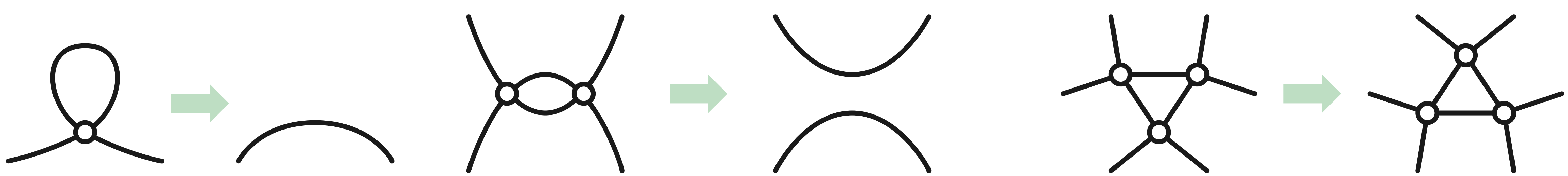
Winding number

- ***wind*(*γ*, *p*)** = number of times *γ* winds ccw around *p*
  - At points *p* not on *γ*, given by *Alexander numbering*: *[Meister 1770][Möbius 1865][Alexander 1928]*
  - At vertices, average the winding numbers of all four incident faces



Winding Lemma

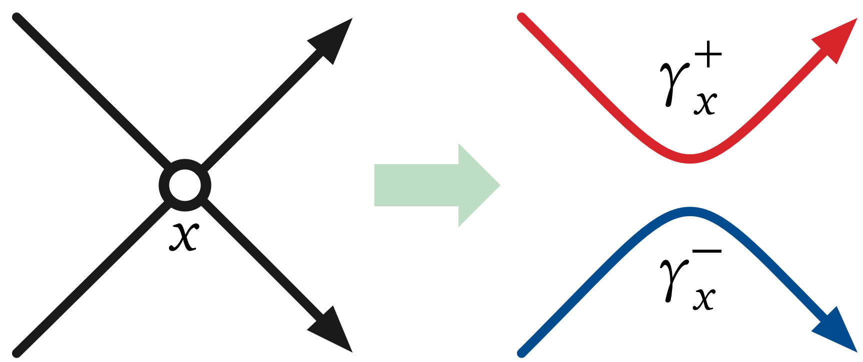
- Each 3→3 move changes the winding numbers of exactly three vertices, each by exactly 1.
- 1→0 and 2→0 moves do not change the winding numbers of any vertices.



1→0 2→0 3→3

Vertex types

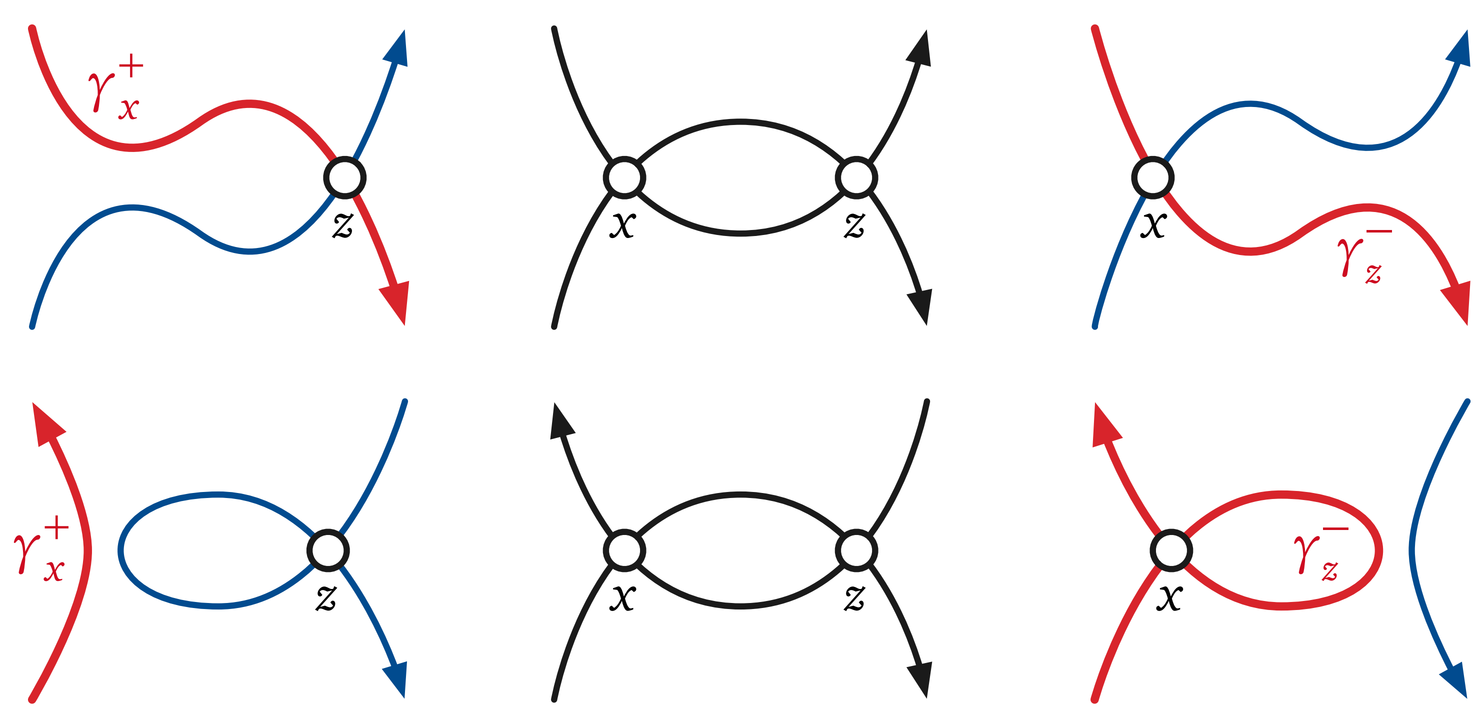
- Consider a contractible closed curve γ in Σ.
- Smoothing γ at any vertex x yields two curves γ_x^+ and γ_x^-.



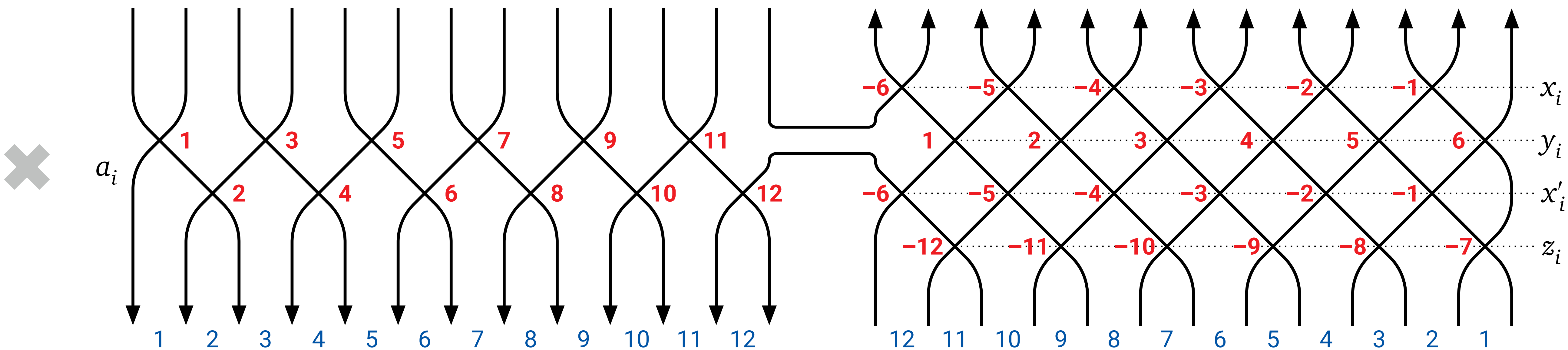
- Define *type*(x) := *wind*(γ_x^+, o)
- Vertices x and z *are* ***complementary*** if *type*$(x) = -$*type*(z).

Type Lemma

- No homotopy move changes the type of any vertex.
- Every 1→0 move deletes a vertex of type 0.
- Every 2→0 move deletes two complementary vertices.

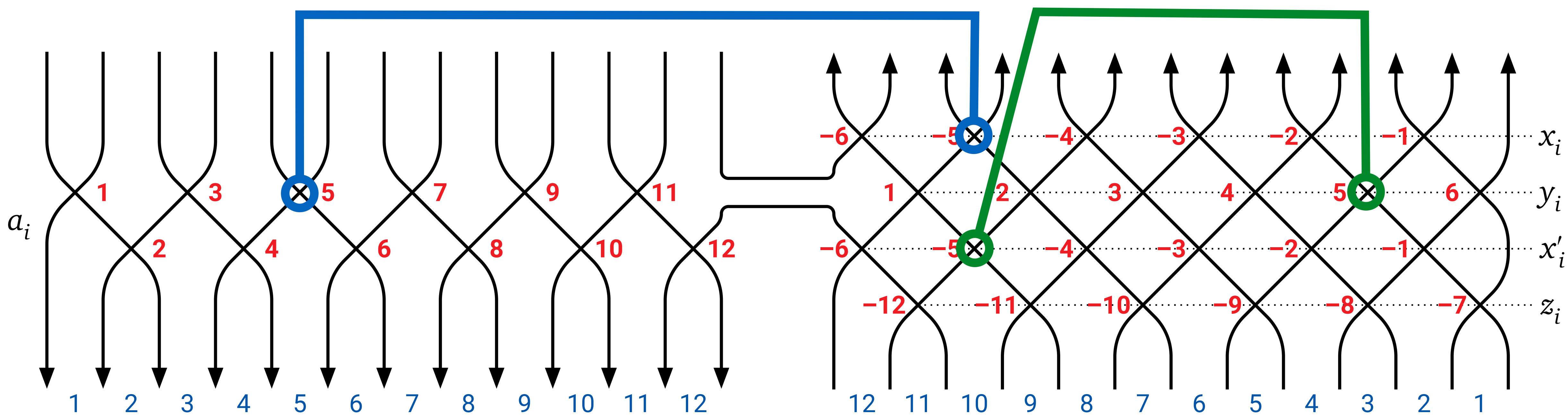


Vertex matching



- Every homotopy that contracts the bad curve defines a matching between complementary vertices.
- In every such matching for this curve, the differences of winding numbers sum to $\boldsymbol{\Omega(n^2)}$

Vertex matching



- Every homotopy that contracts the bad curve defines a matching between complementary vertices.
- In every such matching for this curve, the differences of winding numbers sum to **$\Omega(n^2)$**

Higher-Genus Upper Bound

ISRAEL JOURNAL OF MATHEMATICS, Vol. 51, Nos. 1–2, 1985

INTERSECTIONS OF CURVES ON SURFACES

BY

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ABSTRACT

The authors consider curves on surfaces which have more intersections than the least possible in their homotopy class.

THEOREM 1. *Let f be a general position arc or loop on an orientable surface F which is homotopic to an embedding but not embedded. Then there is an embedded 1-gon or 2-gon on F bounded by part of the image of f.*

THEOREM 2. *Let f be a general position arc or loop on an orientable surface F which has excess self-intersection. Then there is a singular 1-gon or 2-gon on F bounded by part of the image of f.*

Examples are given showing that analogous results for the case of two curves on a surface do not hold except in the well-known special case when each curve is simple.

Let C_1 and C_2 be simple closed curves on the annulus A. It is easy to show that if C_1 and C_2 intersect and do so transversely, then there must be a 2-disc D in A whose boundary is $\lambda_1 \cup \lambda_2$ where λ_i is a sub-arc of C_i. We call such a disc a *2-gon* between C_1 and C_2. If two simple closed curves C_1 and C_2 on a surface F intersect transversely, we will say that C_1 and C_2 have *excess intersection* if one of them can be homotoped so as to reduce the number of intersection points with the other. The natural generalisation of the above result about two curves on the annulus is that if C_1 and C_2 are simple closed curves on a surface F and if they have excess intersection then there is a 2-gon between C_1 and C_2. This result is fairly well known, but, for completeness, we give a proof at the start of §3.

In this paper, we consider the question of finding analogous results about the intersection of two possibly singular loops on a surface and about the self-intersection of a single loop. Various results in this area have been assumed to be obvious by some authors. However, we give examples which demonstrate that

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Making Curves Minimally Crossing by Reidemeister Moves

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Received February 22, 1996

DEDICATED TO PROFESSOR W. T. TUTTE ON THE OCCASION OF HIS EIGHTIETH BIRTHDAY

Let $C_1, \ldots, C_k$ be a system of closed curves on a triangulizable surface S. The system is called *minimally crossing* if each curve C_i has a minimal number of self-intersections among all curves C_i' freely homotopic to C_i and if each pair C_i, C_j has a minimal number of intersections among all curve pairs C_i', C_j' freely homotopic to C_i, C_j respectively ($i, j = 1, \ldots, k, i \neq j$). The system is called *regular* if each point traversed at least twice by these curves is traversed exactly twice, and forms a crossing.

We show that we can make any regular system minimally crossing by applying Reidemeister moves in such a way that at each move the number of crossings does not increase. It implies a finite algorithm to make a given system of curves minimally crossing by Reidemeister moves. © 1997 Academic Press

1. INTRODUCTION AND FORMULATION OF THE THEOREM

Let S be a surface. A *closed curve* on S is a continuous function $C: S^1 \to S$ (where S^1 is the unit circle in the complex plane). Two closed curves C and C' are *freely homotopic*, in notation: $C \sim C'$, if there exists a continuous function $\Phi: S^1 \times [0, 1] \to S$ such that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C'(z)$ for all $z \in S^1$.

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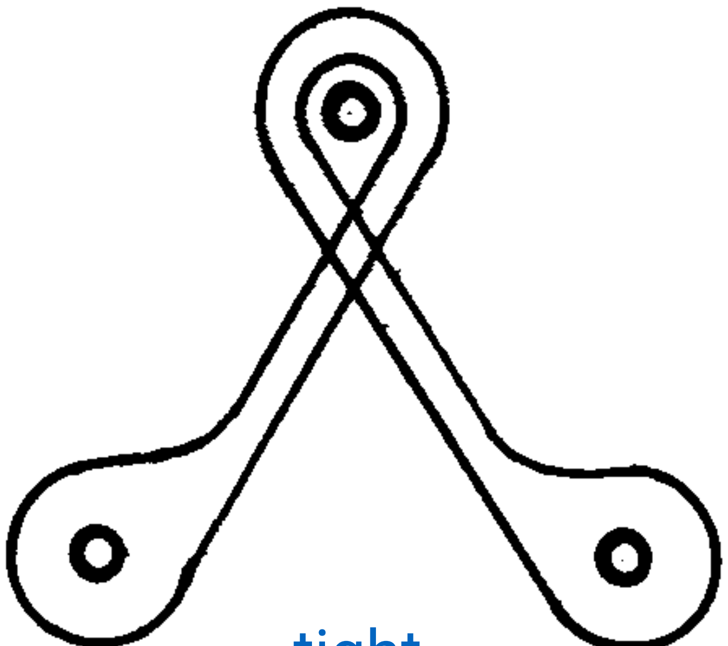
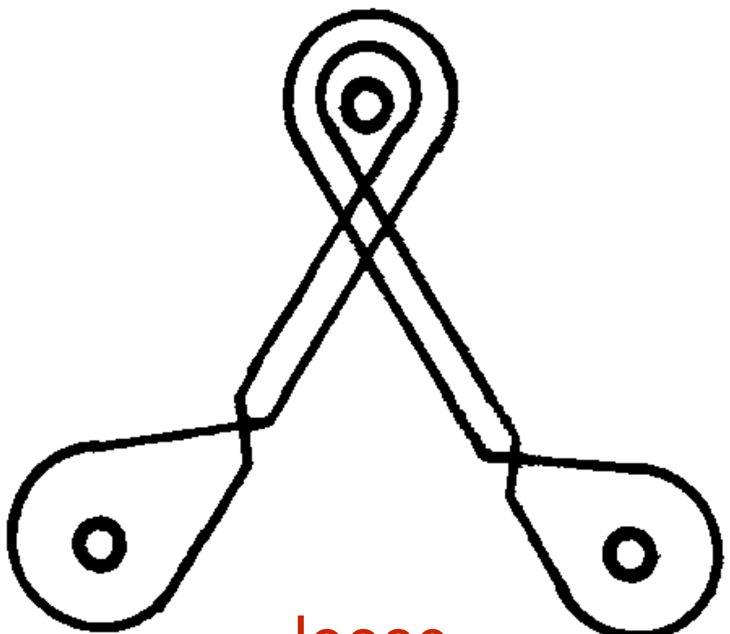
Tightening curves on surfaces

- A curve is ***tight*** if no homotopic curve has fewer vertices, and *loose* otherwise.



tight



loose

[Hass Scott 1994]

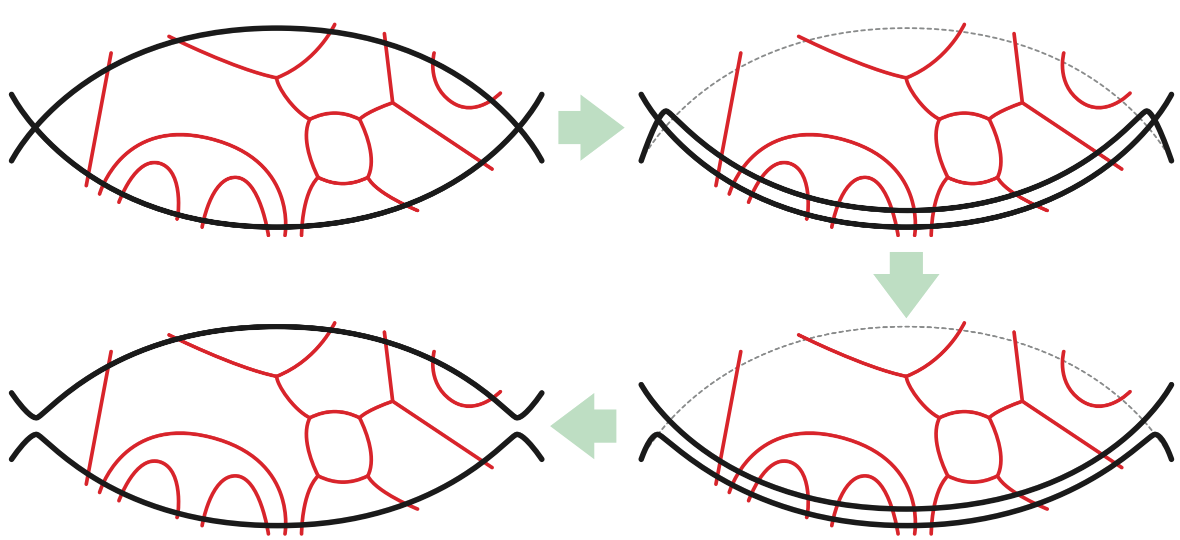
Tightening curves on surfaces

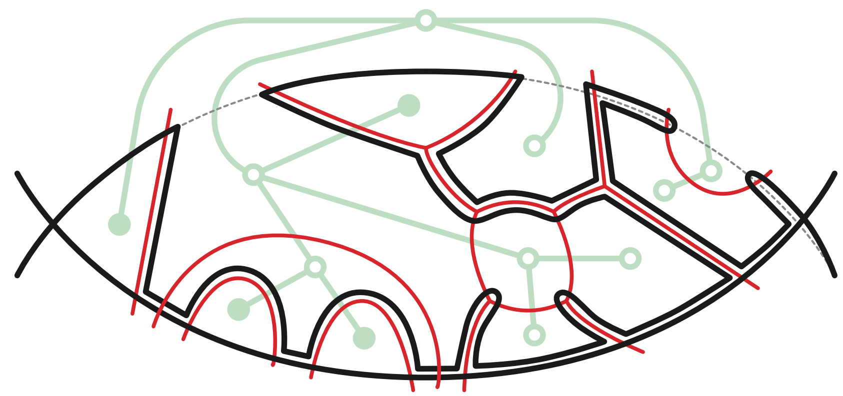
- A curve is ***tight*** if no homotopic curve has fewer vertices.
- **Theorem:** Any closed curve on any surface can be *tightened* using a *finite* number of *non-increasing* homotopy moves.
  [Grayson 1989] [Angenent 1988,1991] [Hass Scott 1994] [de Graff, Schrijver 1997] [Paterson 2002]
- $O(n^2)$ moves suffice for curves on the torus, or if the tightened curve is simple *[Hass Scott 1994]*
- $n^{O(g)}2^{O(n)}$ moves suffice in general.
  - Follows from counting 4-regular n-vertex genus-g surface maps. *[Bender Canfield 1986]*

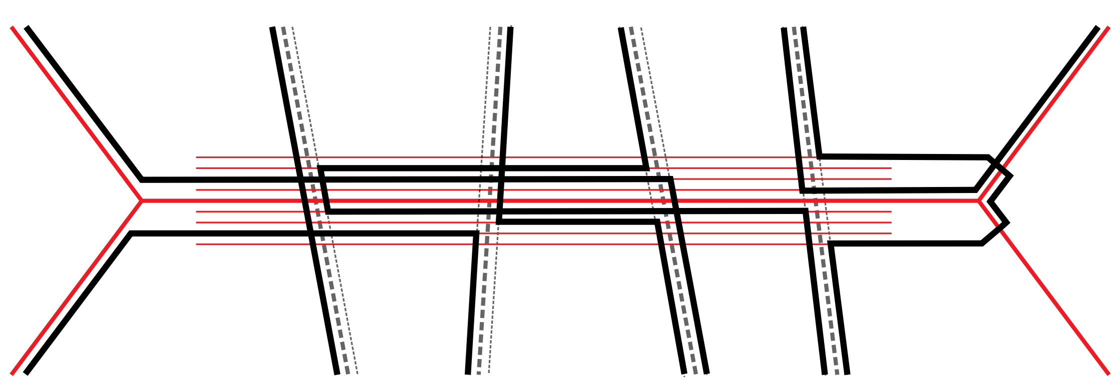
New upper bound

[Chang Erickson Letscher deMesmay Schleimer Sedgwick Thurston Tillmann 2018]

- **Theorem:** Any closed curve with n vertices on any orientable surface can be tightened using $O(n^4)$ homotopy moves.



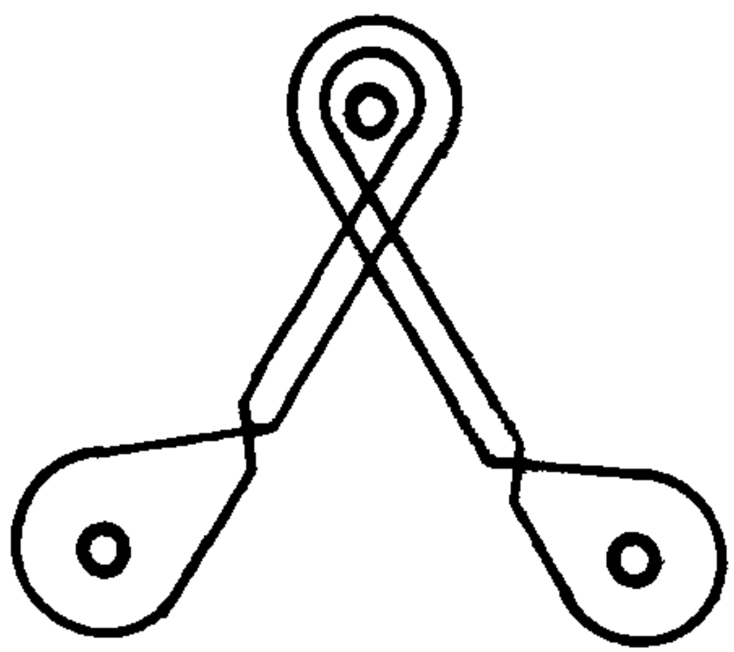




Steinitz doesn't work

[Hass Scott 1994]

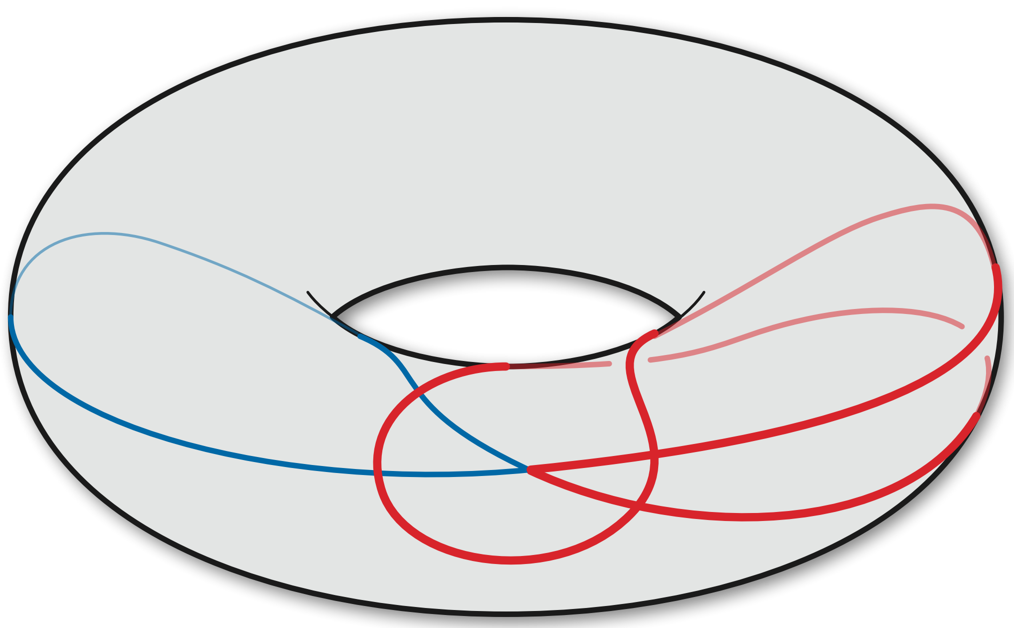
- Non-minimal curves on surfaces do ***not*** necessarily contain monogons or bigons.

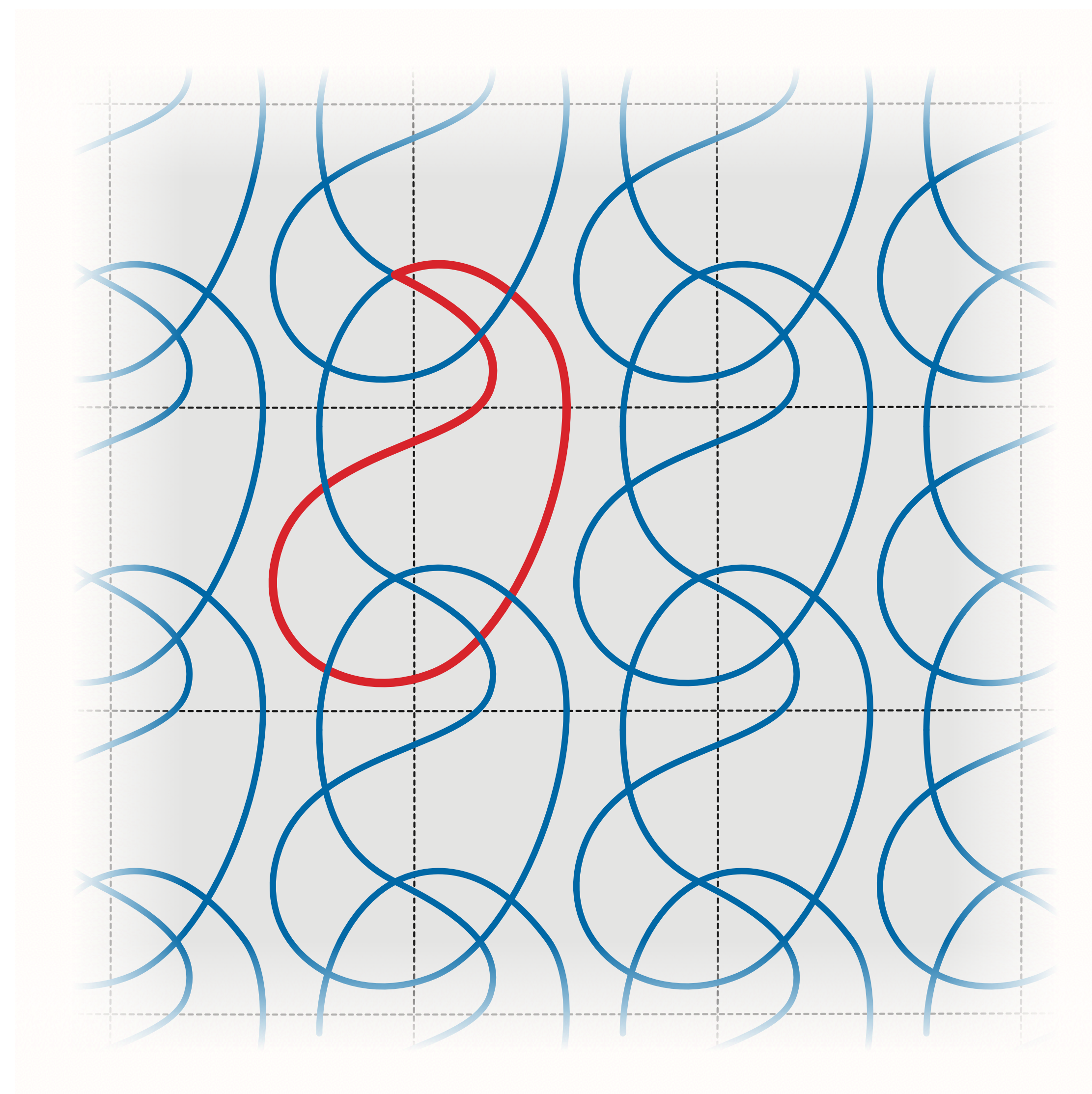


Basic singular monogons

[Hass Scott 1994]

- A *singular monogon* is a contractible subloop
- A singular monogon is *basic* if it lifts to an *embedded* loop in the universal cover.

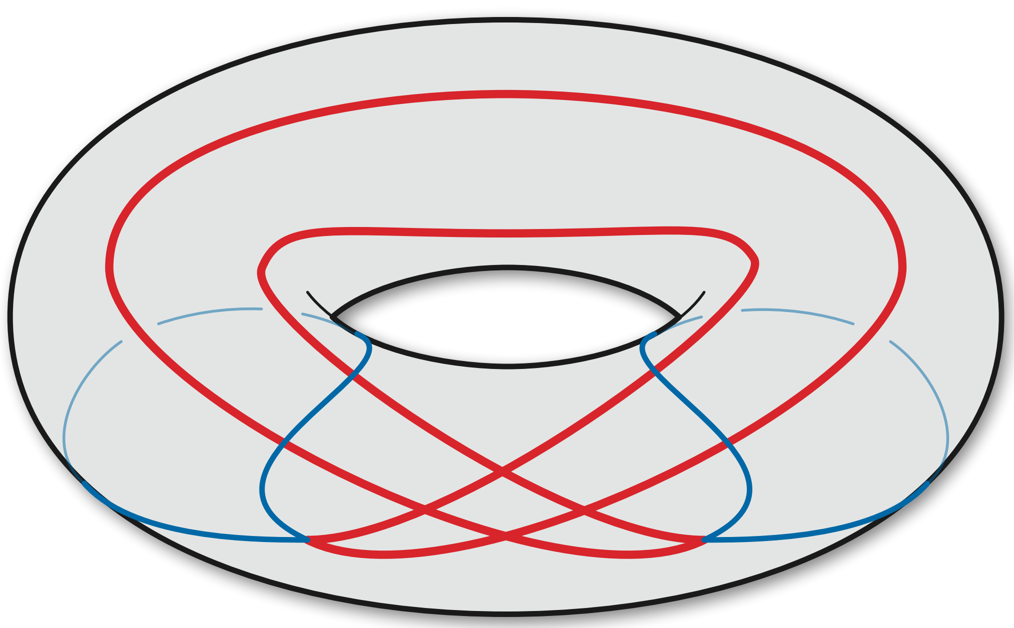


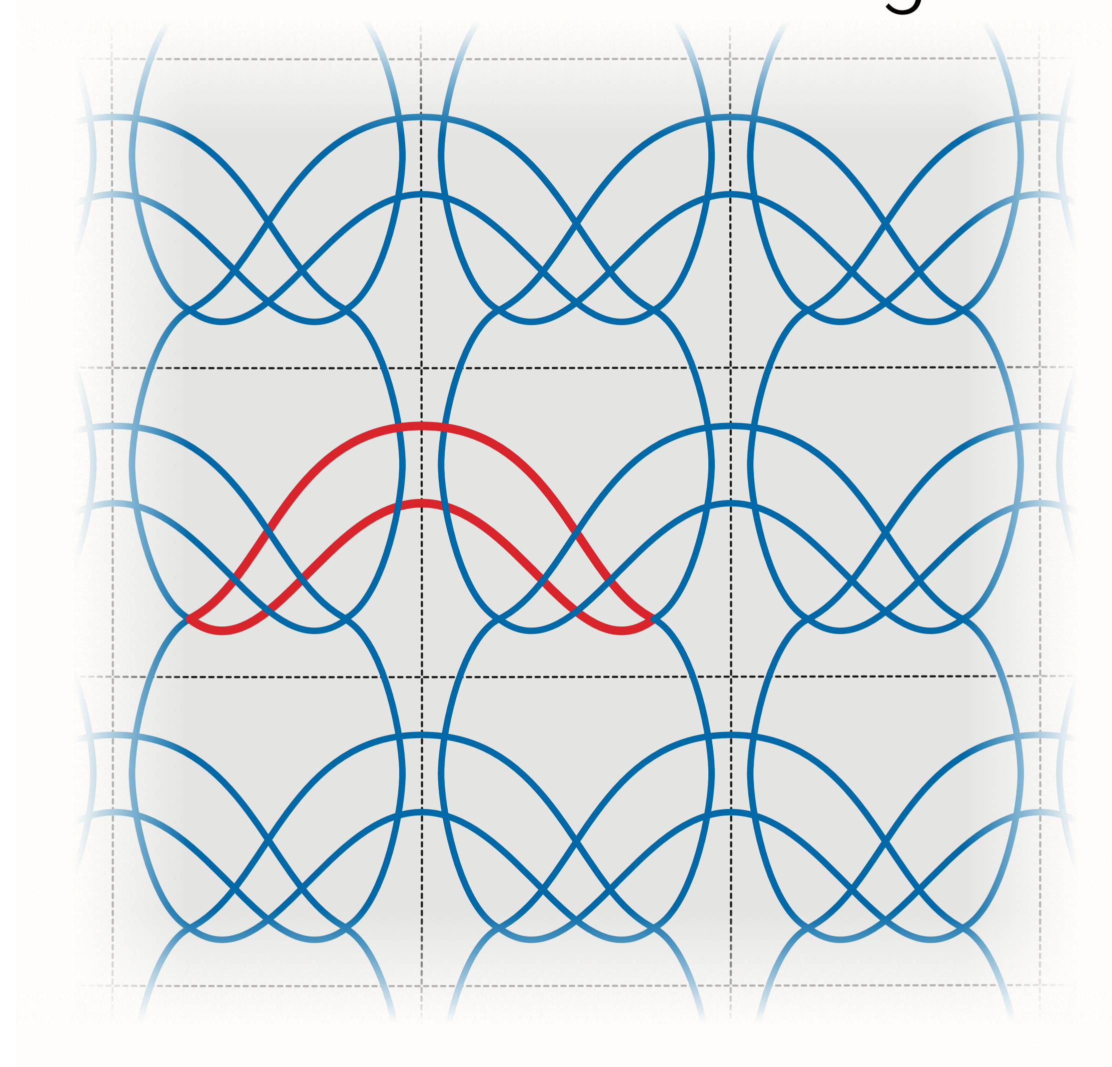


Basic singular bigons

[Hass Scott 1994]

- A *singular bigon* is a homotopic pair of non-overlapping subpaths
- A singular bigon is *basic* if it lifts to an *embedded* bigon in the universal cover

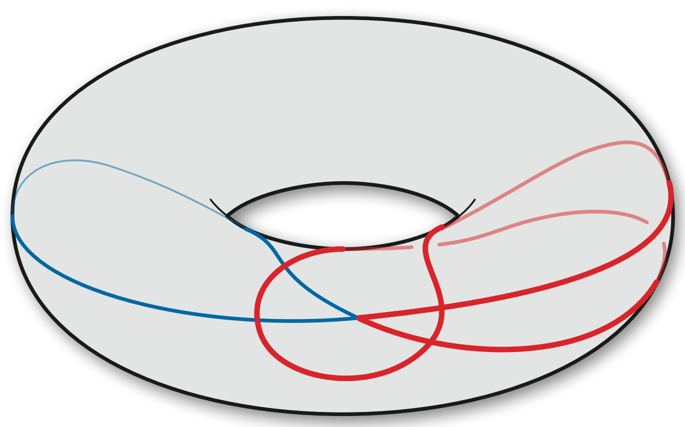


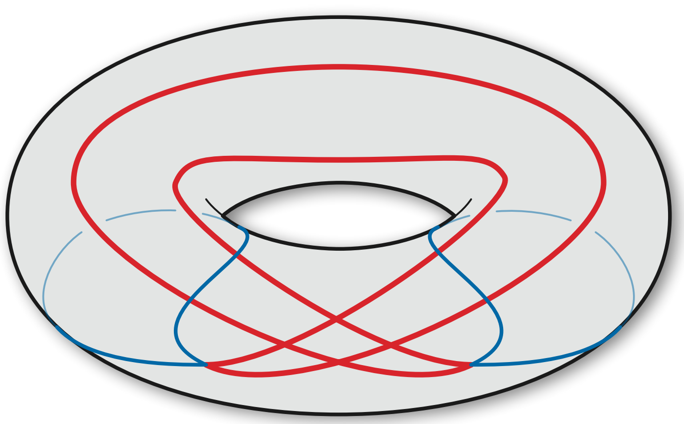


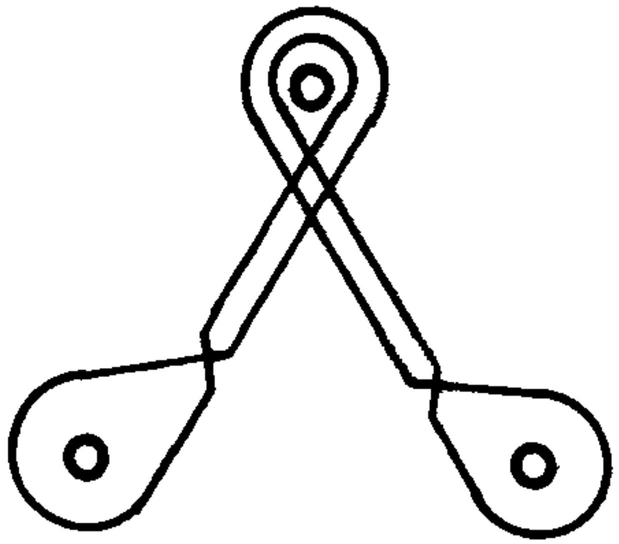
Steinitz almost works

[Steinitz 1916]
[Hass Scott 1994]

- **Lemma:** Every loose curve on the sphere or the torus has either an *embedded* loop or an *embedded* bigon.
- **Lemma:** Every loose curve on any orientable surface has either a *basic singular* monogon or a *basic singular* bigon.

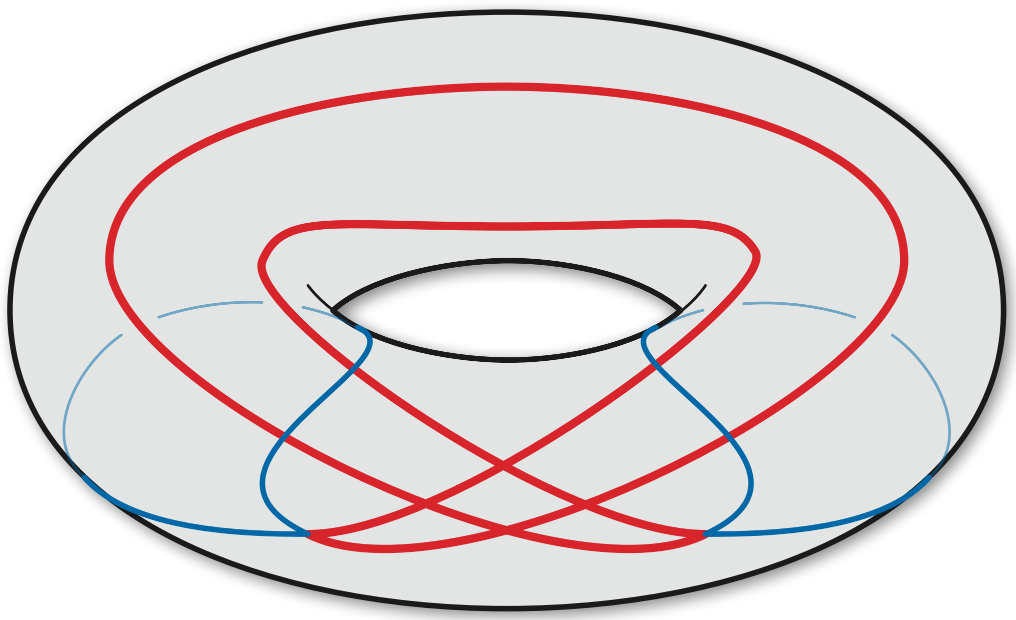


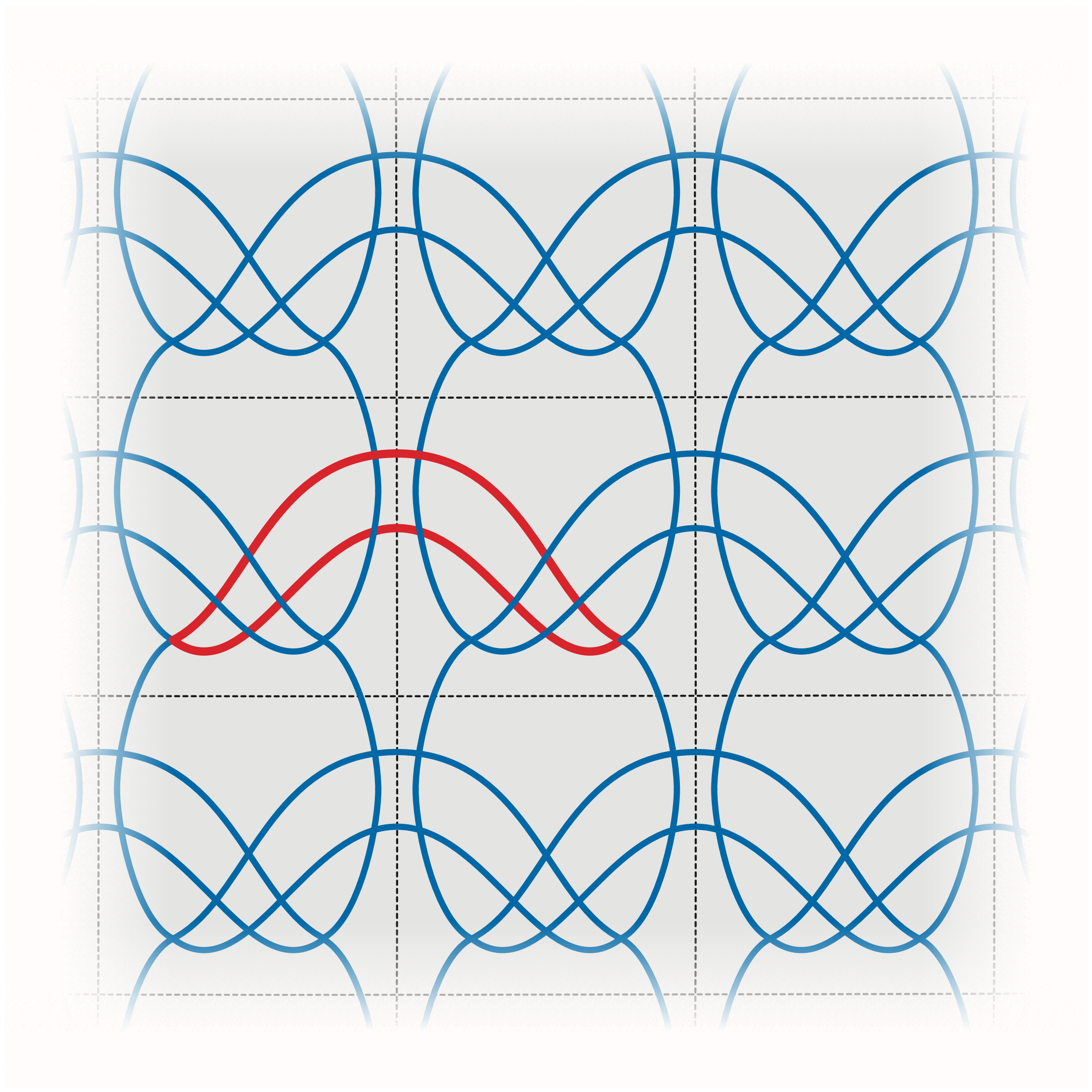




Steinitz still doesn't work

- Removing one face in a basic sungular bigon might add a face somewhere else!





Steinitz still doesn't work

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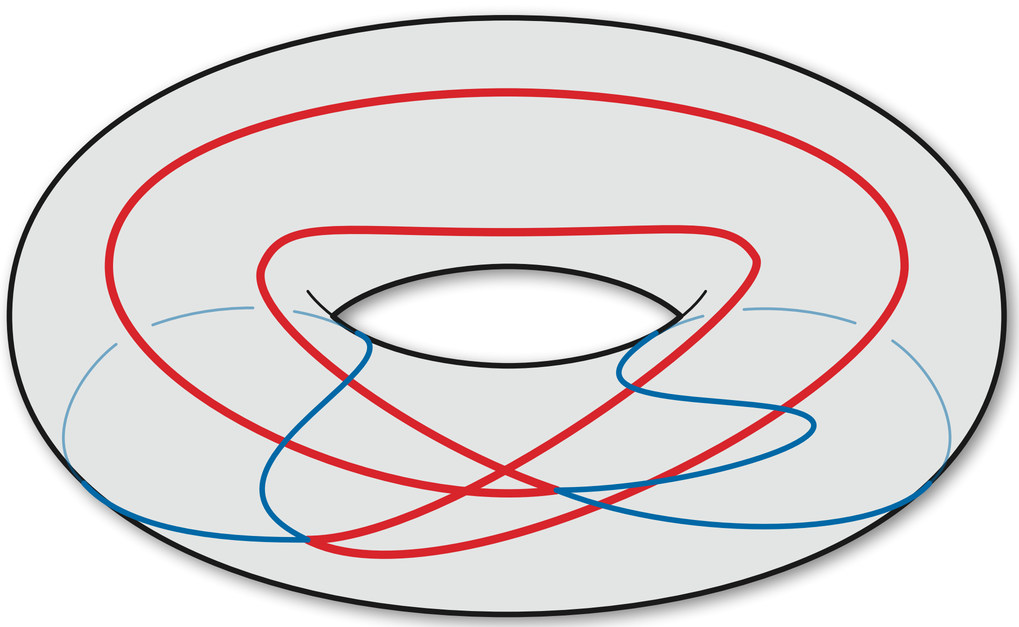
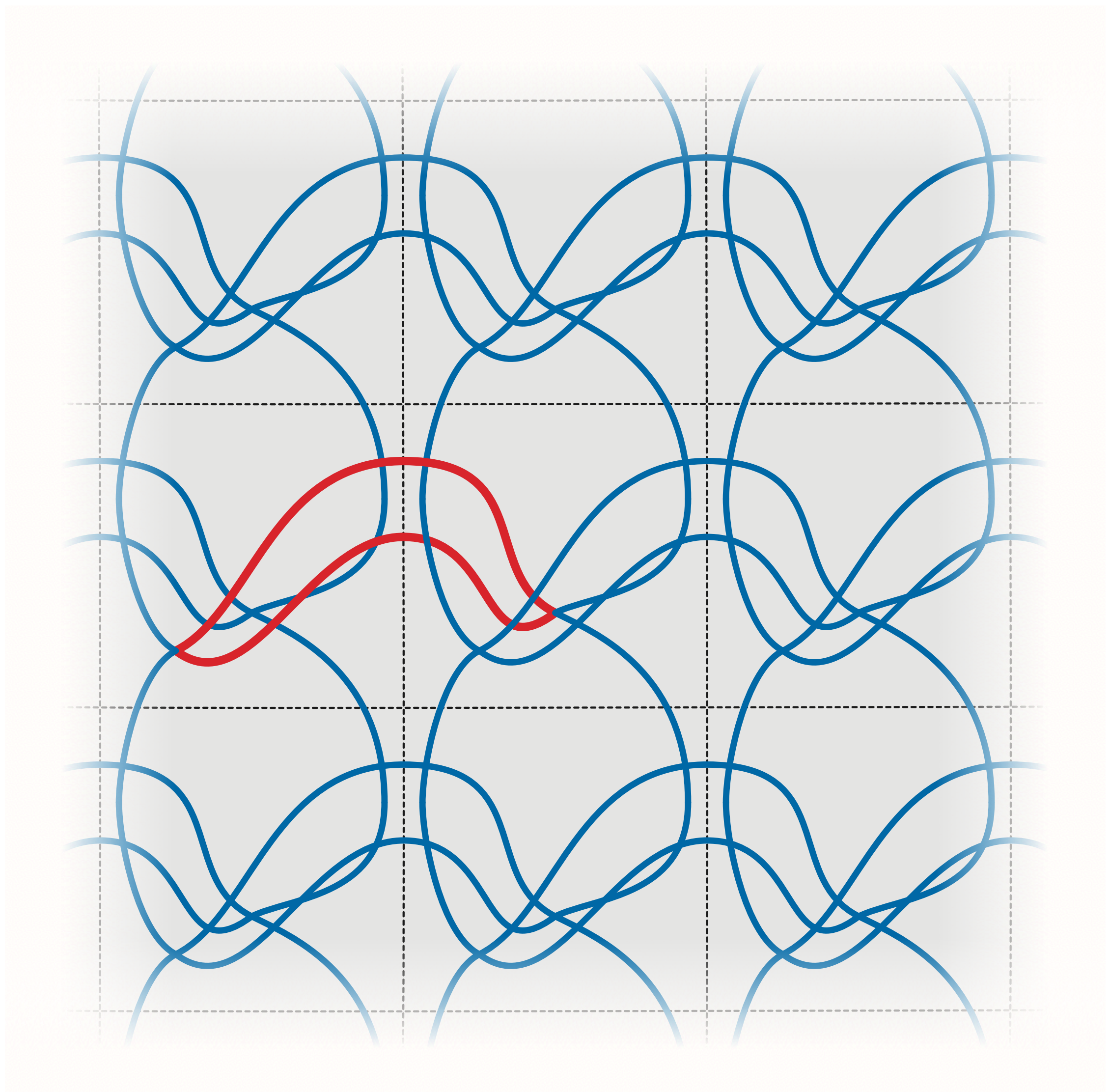
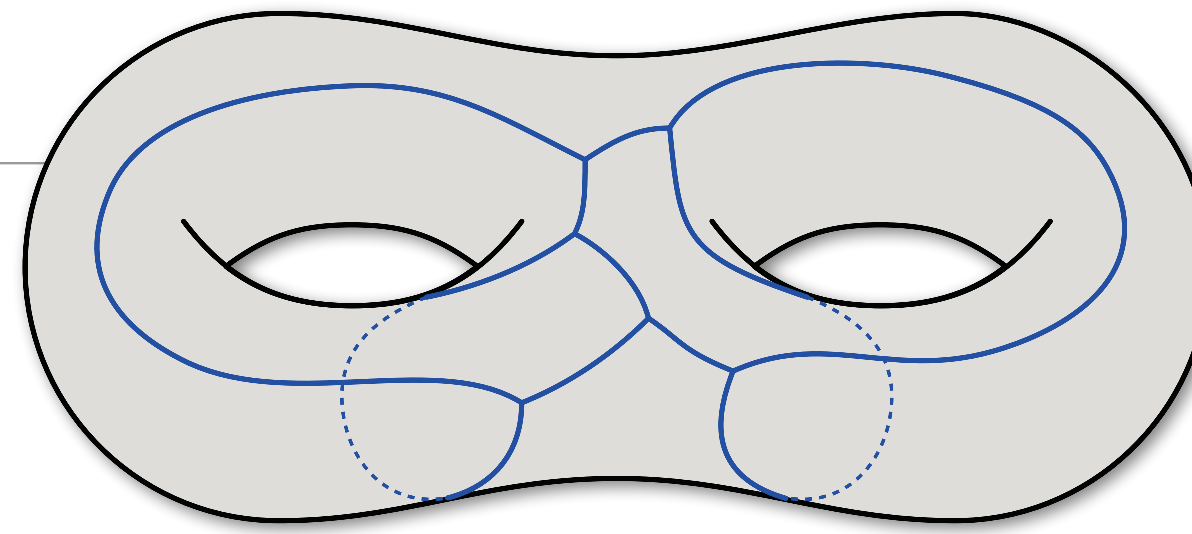
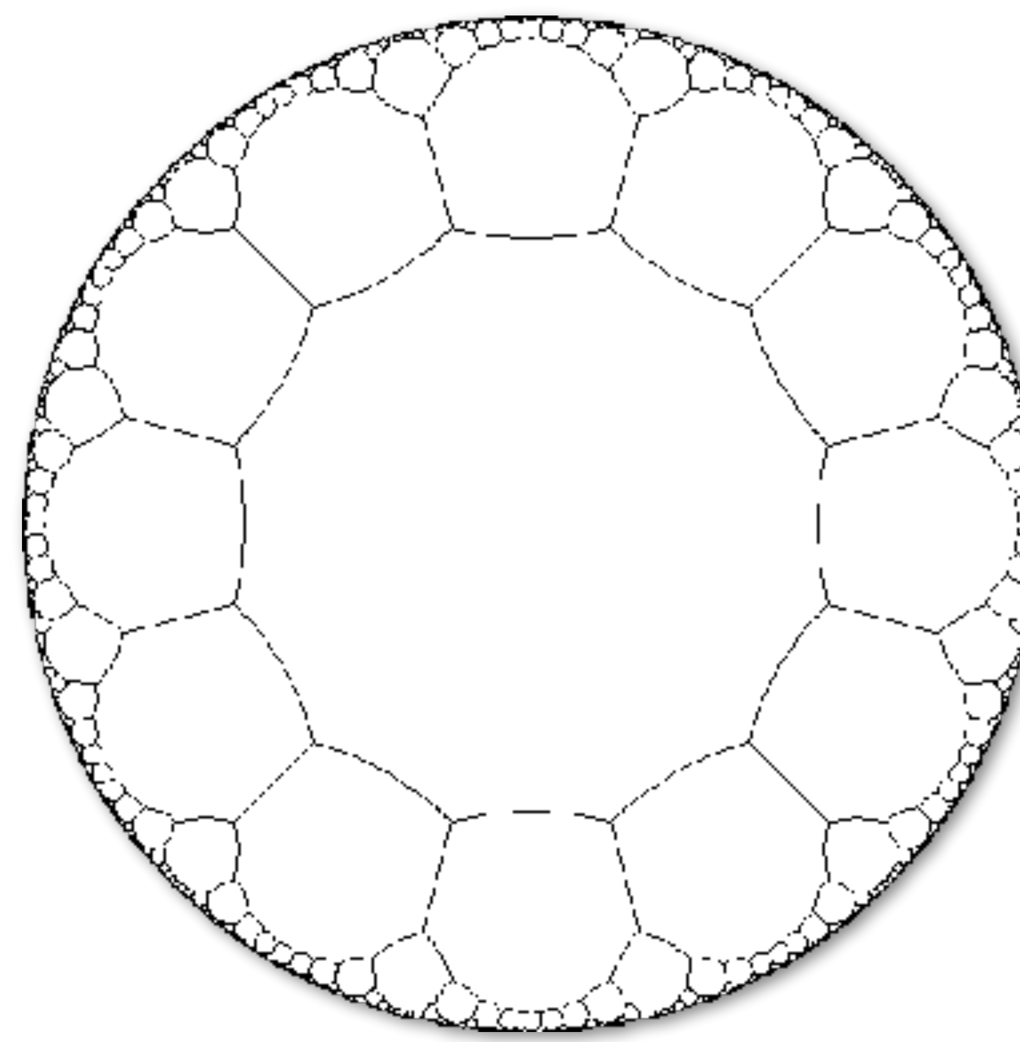




Tiling

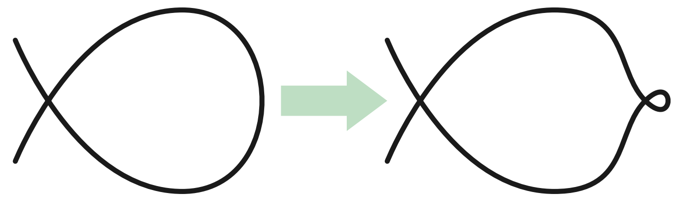
- Let G be a triangulation of the curve γ.
- Choose any tree-cotree decomposition (T, L, C) of G.
- Dual reduced cut locus X = $reduce((C \cup L)^*)$
- Universal cover $\hat{X}$ is a regular trivalent tiling of $6g$-gons

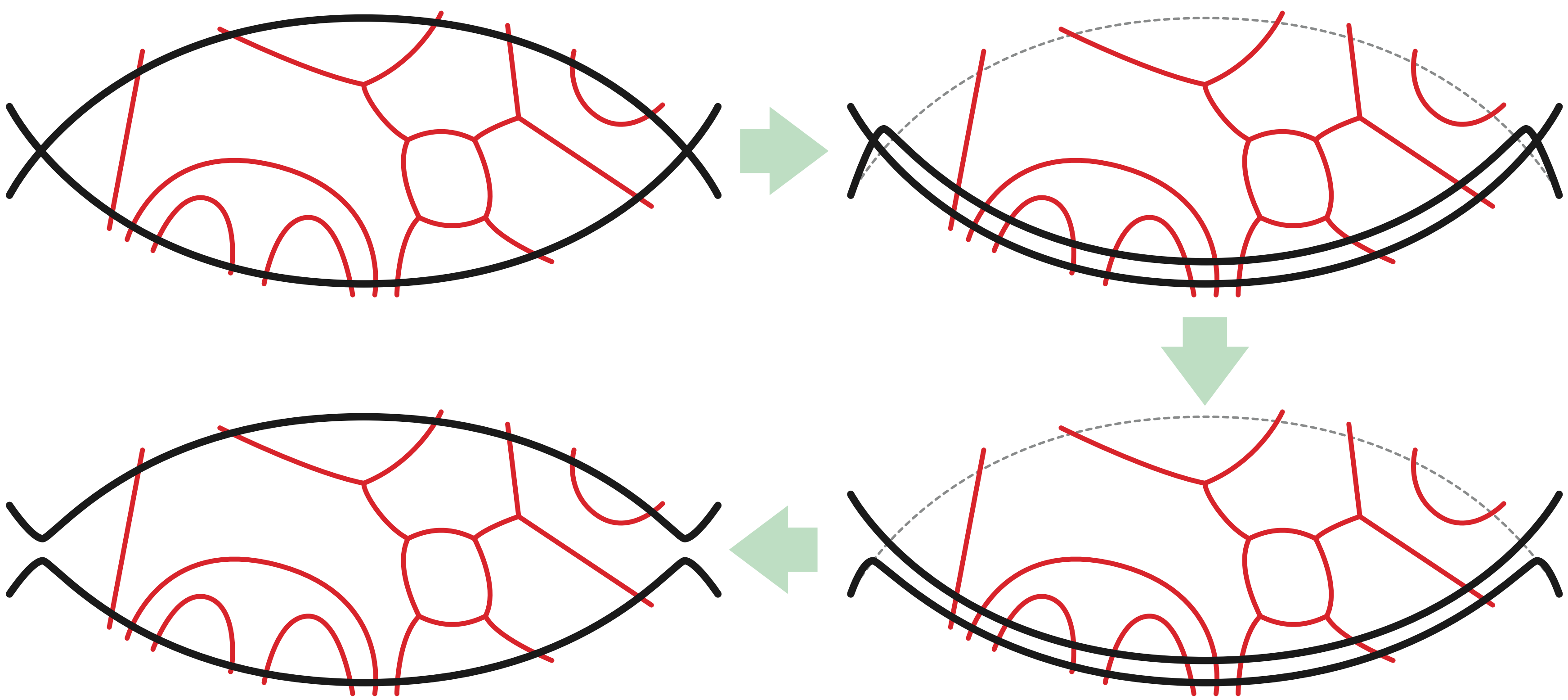




High-level strategy

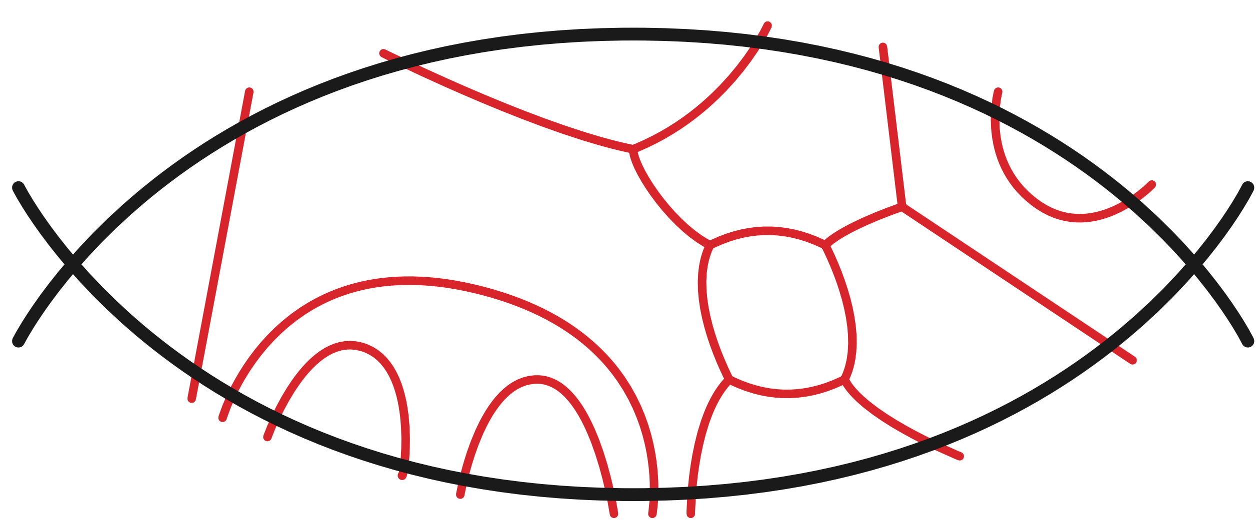
- Repeat until the curve is in minimal position:
  - Find a basic singular (monogon or) bigon β
  - Swap the bounding paths of β as follows:





Bigon “geometry”

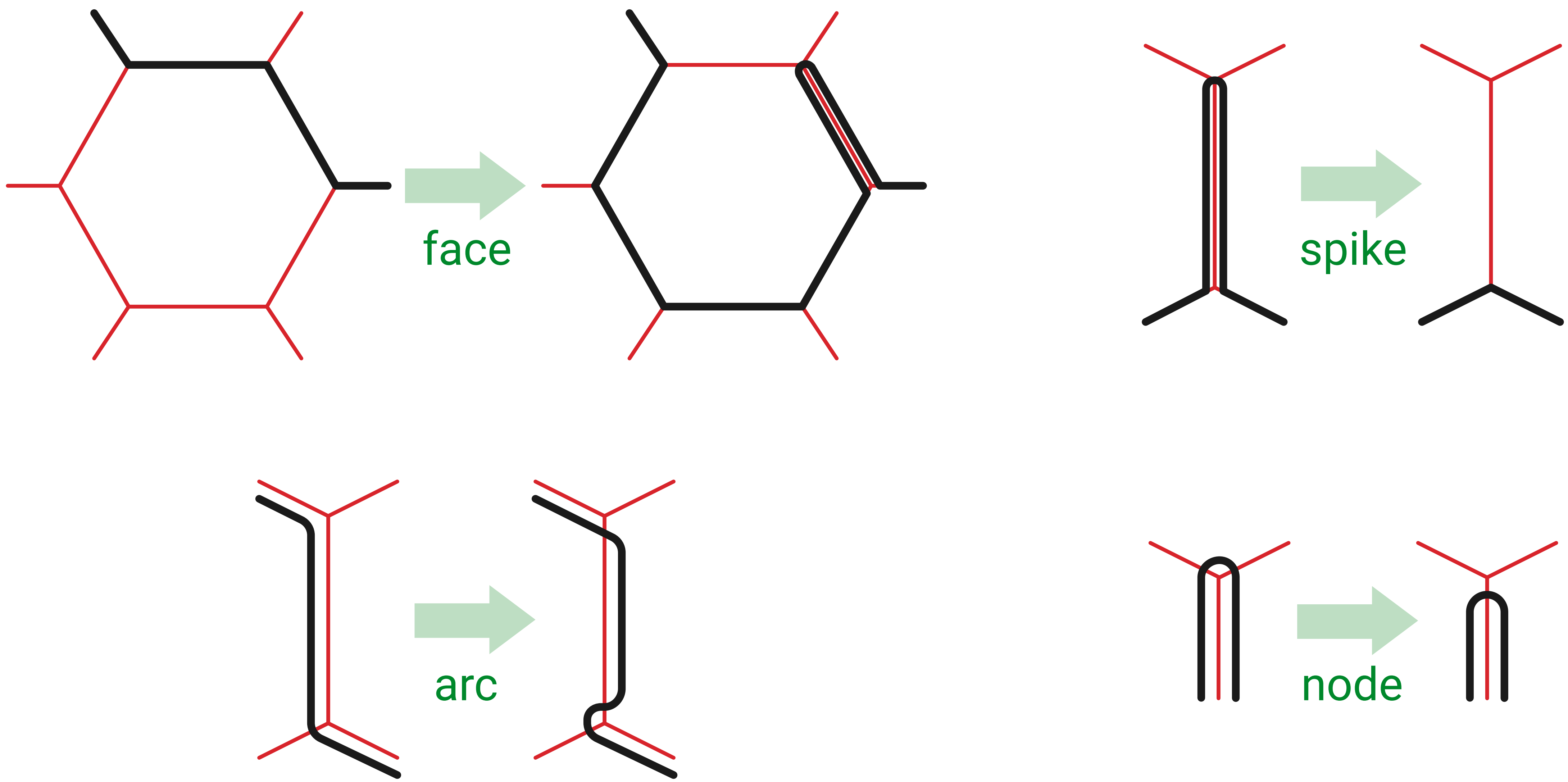
- Curve γ intersects X at most n times
- So any basic singular bigon intersects X at most n times
- Discrete Gauss-Bonnet $\Rightarrow$ interior of lifted bigon $\hat{\beta}$ contains O(n) fragments of tiles in the tiling $\hat{X}$.



Coarse homotopy

[Dehn 1916]

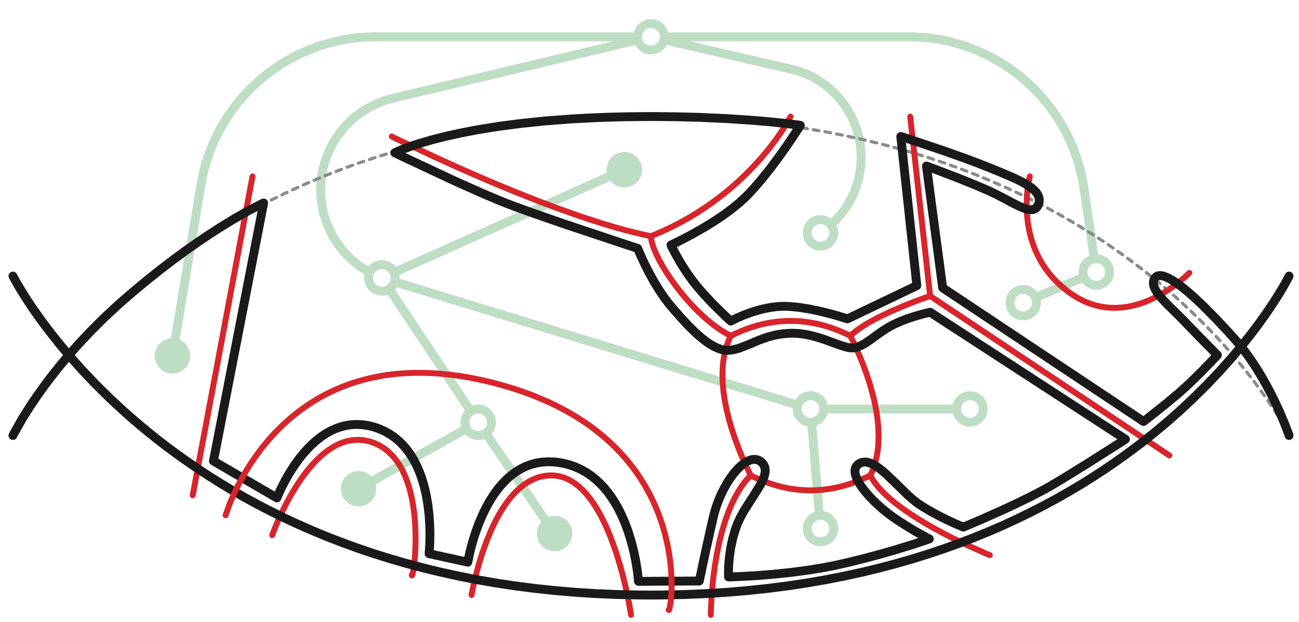
- Move one boundary of the lifted bigon to the other using $O(n)$ "graph homotopy moves":



Coarse homotopy

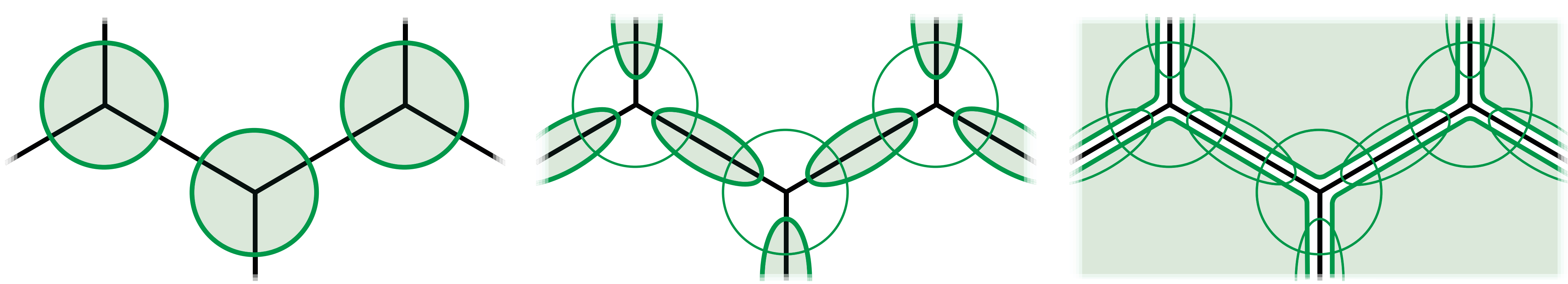
[Dehn 1916]

- Move one boundary of the lifted bigon to the other using O(n) "graph homotopy moves":



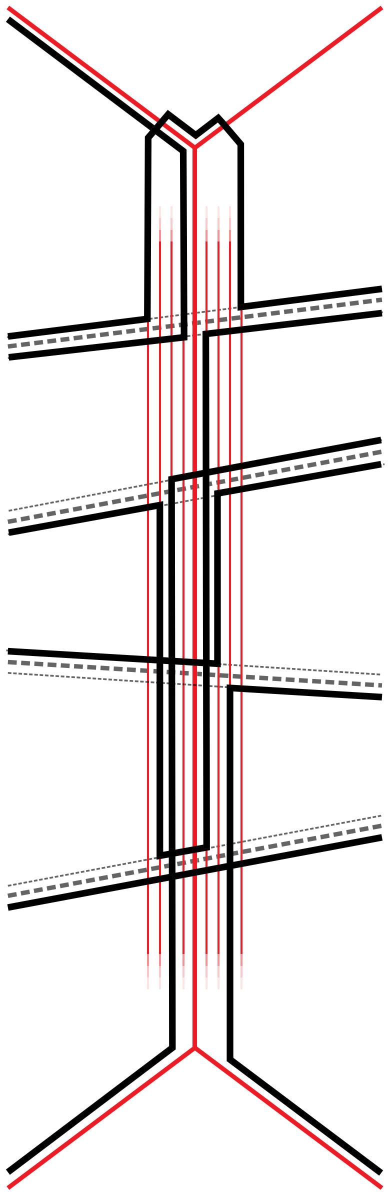
Bubbles

- Cover Σ with open balls:
  - Around each node
  - Around portion of each arc outside the node bubbles
  - Around subset of the face outside the node and arc bubbles
- We perform each "graph move" inside a bubble to avoid interference with translates.



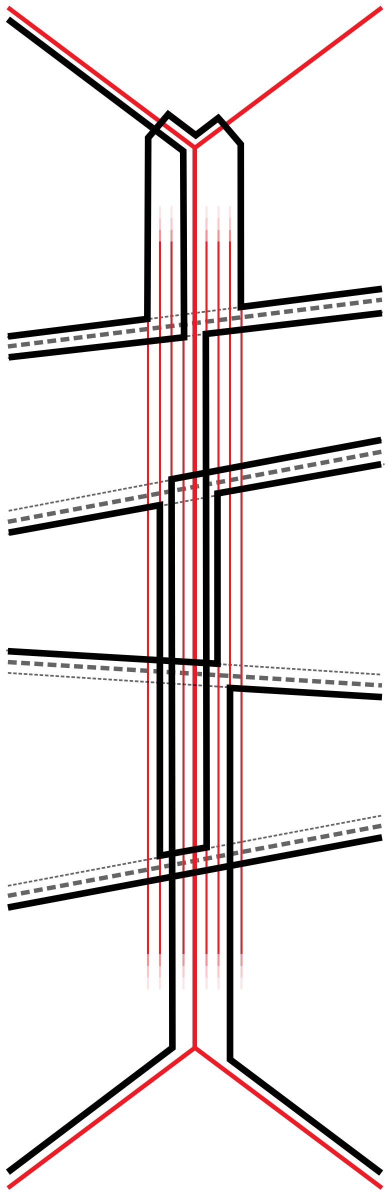
Inside bubbles

- Face bubble: ≤4n vertices
  - doubled curve segments ⇒ quadrupled vertices
- Arc bubbles: O(n^2) vertices each
  - O(n) parallel “tracks” for translates of the moving frontier
  - Crossed by O(n) curve segments
- Node bubbles: O(n^2) vertices each
  - Exchange between three incident sets of tracks



Graph moves

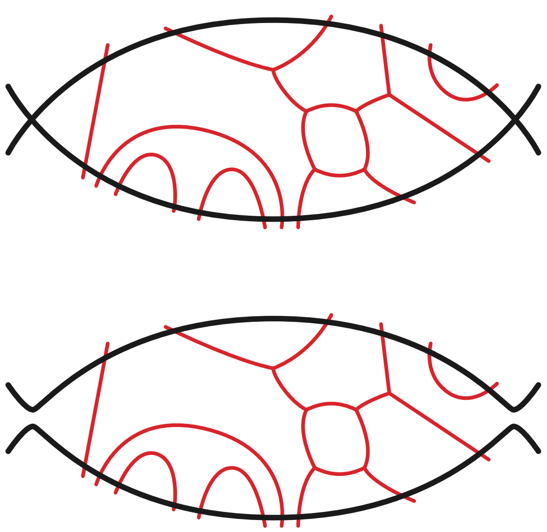
- Face move ⇒ $O(n)$ homotopy moves
- Spike move ⇒ $O(n^2)$ homotopy moves
- Arc move ⇒ $O(n^2)$ homotopy moves
- Vertex move ⇒ $O(n^2)$ homotopy moves

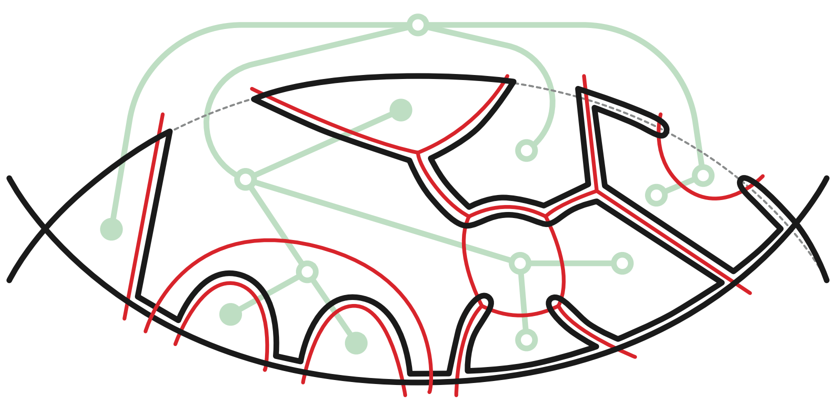


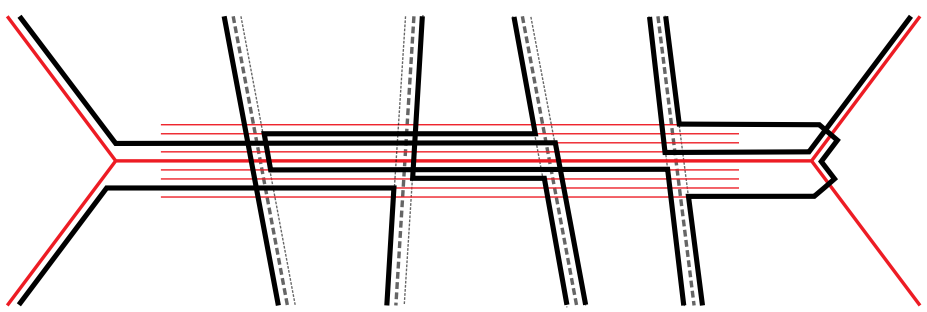
Theorem

Any closed curve with n vertices, on any orientable surface, can be tightened using $O(n^4)$ homotopy moves.

- To simplify the curve, we need to resolve $O(n)$ singular bigons.
- We resolve each singular bigon using $O(n)$ graph moves.
- We implement each graph move using $O(n^2)$ homotopy moves.







Improvements

- **Theorem:** Any closed curve with n vertices, on any orientable surface with genus g, can be tightened using $O(gn^3 \log^2 n)$ homotopy moves.
  - Choose a coarse homotopy with small ***homotopy height***.
    [Har-Peled Nayyeri Salavatipour Sidiropulos 2016]

- **Theorem:** Any closed curve with n vertices, on any orientable surface with genus g *and b>0 boundaries,* can be tightened using $O((g+b)n^3)$ homotopy moves.
  - Use a ***greedy system of arcs*** instead of a reduced cut graph.
  - Dual of the universal-cover tiling is a tree.

Open problems

- What about *non*-orientable surfaces?
  - There may be no singular monogons or bigons.
  - $O(n^2)$ moves suffice in the projective plane or Möbius band.
    - Can we improve projective plane to $O(n^{3/2})$?
  - No other nontrivial upper bounds known.
- What about *multi*curves?
  - No nontrivial upper bounds known (with small exceptions).
- What about *monotone* homotopy?
  - No nontrivial upper bounds known (with small exceptions).
    - But Arnaud and Hsien are getting close!

Summary

Surface	$O(\ )$	$\Omega(\ )$
Sphere or plane	$n^{3/2}$	$n^{3/2}$
Projective plane	n^2	$n^{3/2}$
Sphere with holes	n^2	n^2
Torus	n^2	n^2
Orientable, simple curve	n^2	n^2
Orientable w/bdry	n^3	n^2
Orientable w/o bdry	$n^3 \log^2 n$	n^2
Nonorientable	$\exp(n)$	n^2

Thank you!

Twisting, overlapping coloring of Haken's Gordian Knot, Mick Burton (2015)