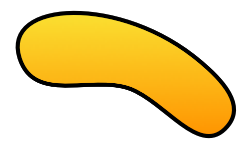
Normal and minimal surfaces

The correspondence between normal surfaces and minimal surfaces has more applications. It can be used to investigate classical problems in Differential Geometry.

Classical Isoperimetric Inequality:

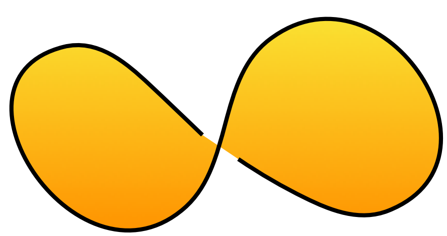
A curve $\boldsymbol{\gamma}$ in $\mathbf{R}^2$ bounds a disk D with $4\pi A \leq L^2$.



Equality holds if and only if $\boldsymbol{\gamma}$ is a circle.

What about if we are given an unknotted curve $\boldsymbol{\gamma}$ in $\mathbf{R}^3$?

Is there a disk with $A \leq f(L)$ for some function f?



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The correspondence between normal surfaces and minimal surfaces has many more applications. It can be used to investigate classical problems in Differential Geometry.

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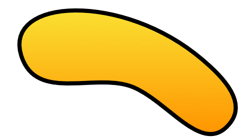
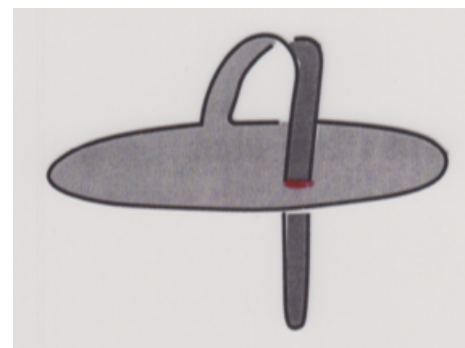


Equality holds if and only if $\boldsymbol{\gamma}$ is a circle.

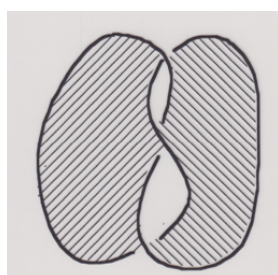
Suppose we are given an unknotted curve $\boldsymbol{\gamma}$ in $\mathbf{R}^3$?

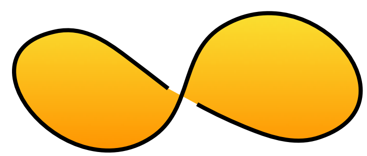
Is there a disk with $A \leq f(L)$ for some function f?



1. There is an immersed disk with $4\pi A \leq L^2$.
   (Andre Weil, 1926)

2. There is an embedded surface with $4\pi A \leq L^2$.
   (W. Blaschke, 1930)

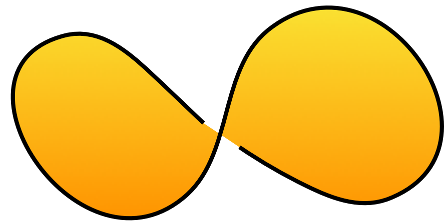




What bounds can we get for an embedded disk?

Normal and minimal surfaces

What bounds can we get on an embedded disk?



Theorem 1. (H-Lagarias-Thurston, 2004)
There is a sequence of unknotted, smooth curves γ_n embedded in R^3, each having length $L = 1$, such that the area of any embedded disk spanning γ_n is greater than n.

Theorem 2. For any embedded closed unknotted smooth curve γ in R^3 having length L and thickness r, there exists a smooth embedded disk of area A, having γ as boundary with

$$A \leq (C_0)^{(L/r)^2} L^2$$

where $C_0 > 1$ is a constant independent of γ, L and r.

For L=1 and thickness r,

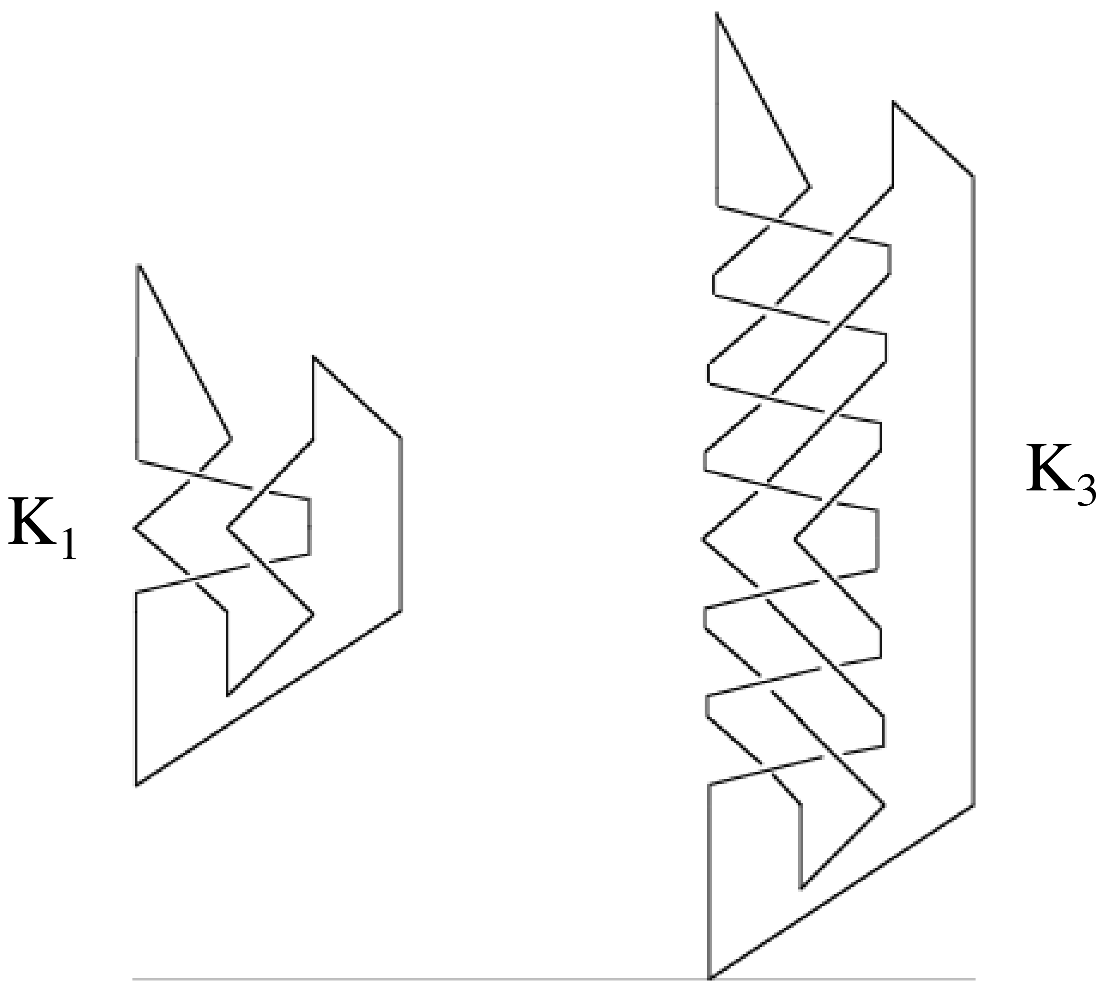
$$A \leq (C_0)^{(1/r)^2}$$

(The thickness of a curve r is the radius of its tubular neighborhood.)

Both results came out of complexity results.

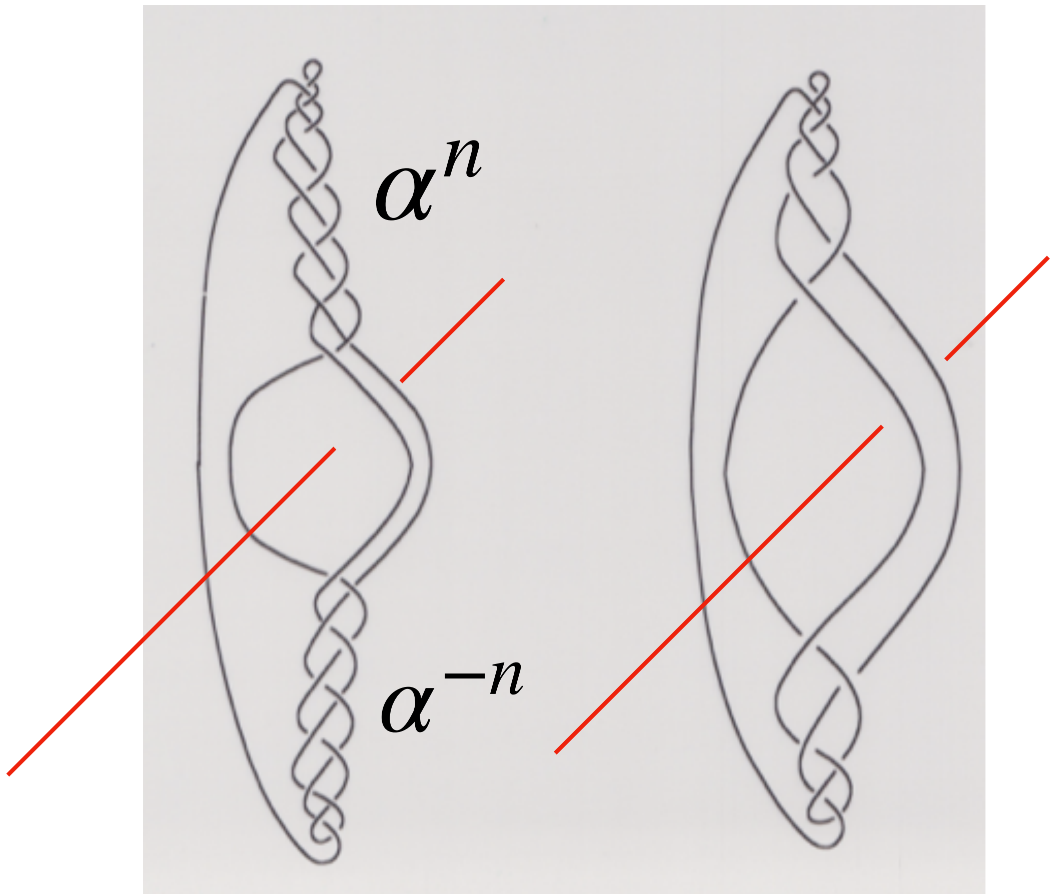
Spanning Disks can be Exponentially Complicated

Theorem (H-Snoeyink-Thurston)
There exists a sequence of unknotted polygons K_n with 11n edges such that any disk spanning the unknot K_n contains at least 2^n triangular faces.



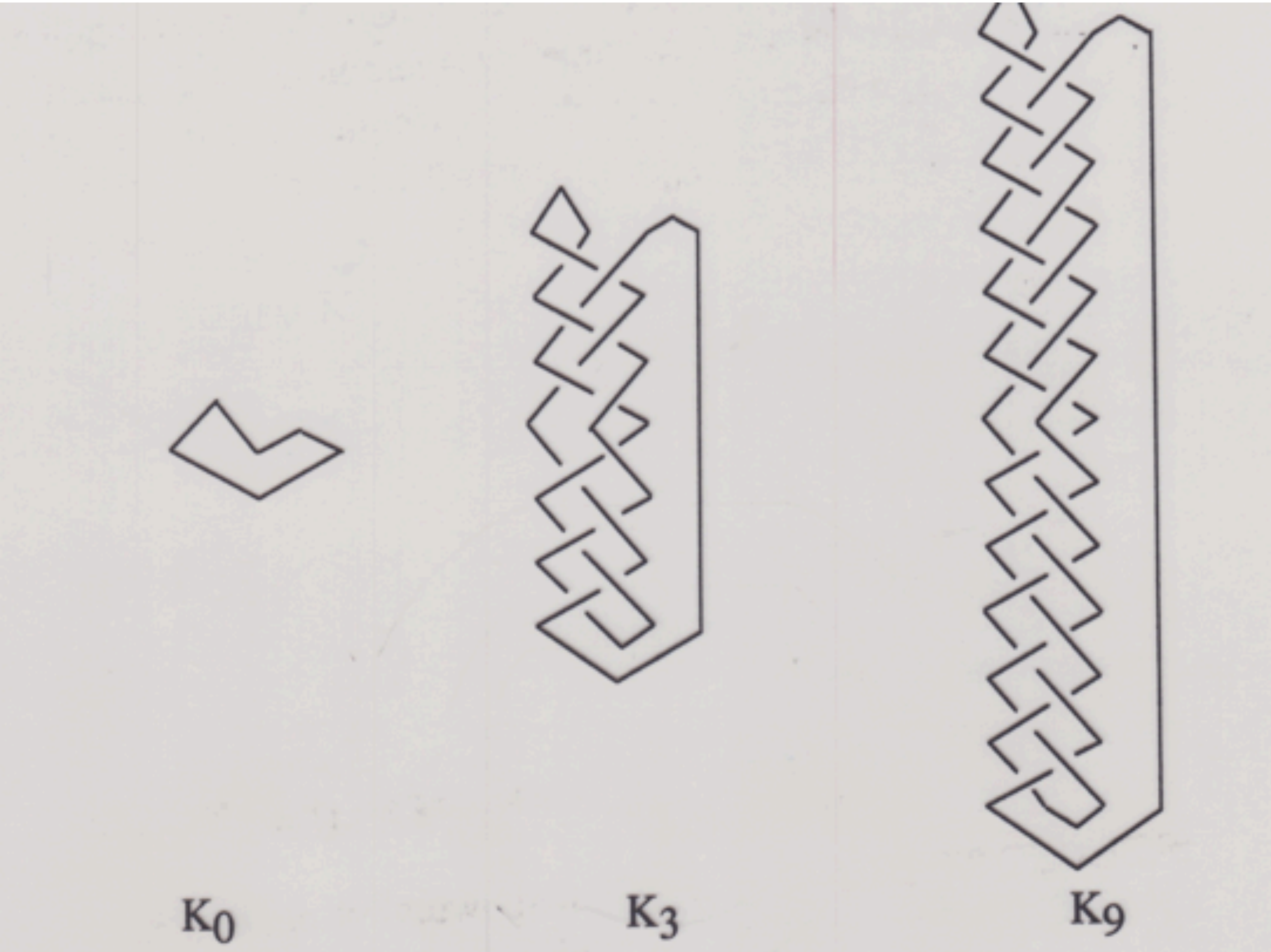
The Idea:

Any disk with boundary K_n must have at least 2^n triangles, since such a disk must cross the red line at least 2^n times, and each triangle intersects a line at most once.



Proof:

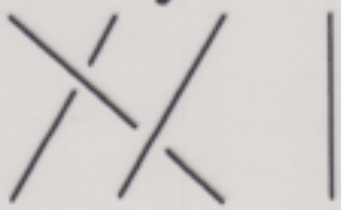
Three curves in the sequence of unknots K_n



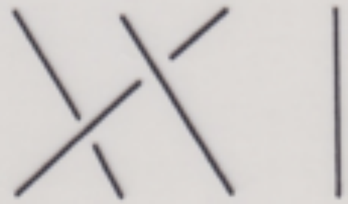
How to construct K_n

Obtained by iterating the braid α n times, followed by n iterations of α^{-1}.

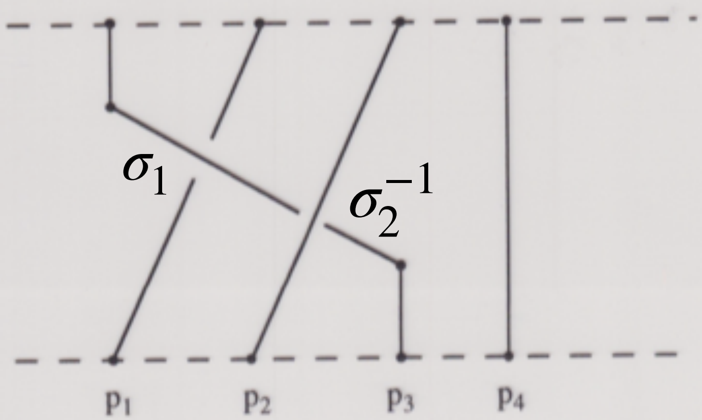
$\alpha =$



$\alpha^{-1} =$

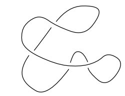


To construct K, start with this braid on four strings

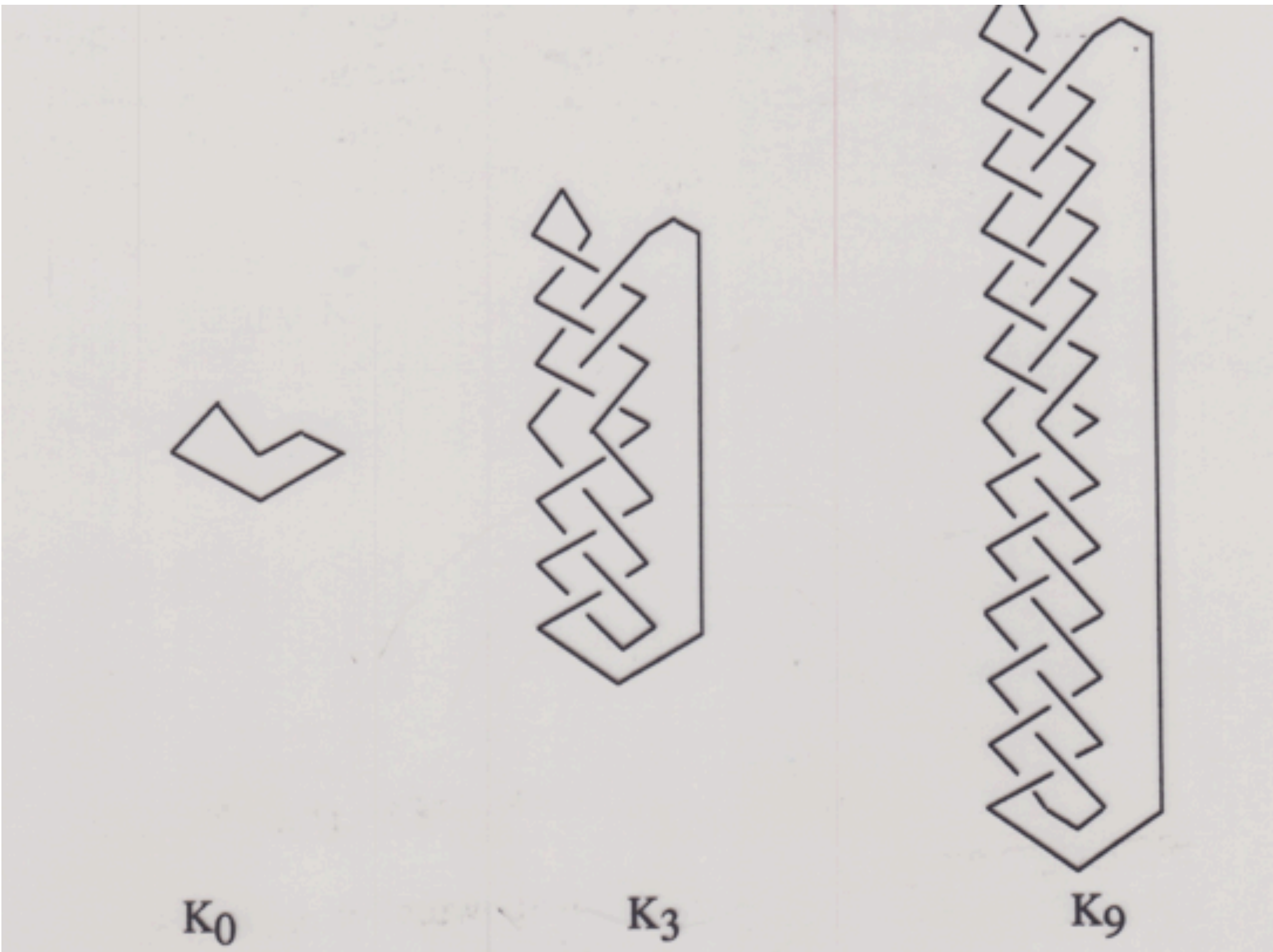


$$\alpha = \sigma_1 \sigma_2^{-1}$$

K_n is obtained by

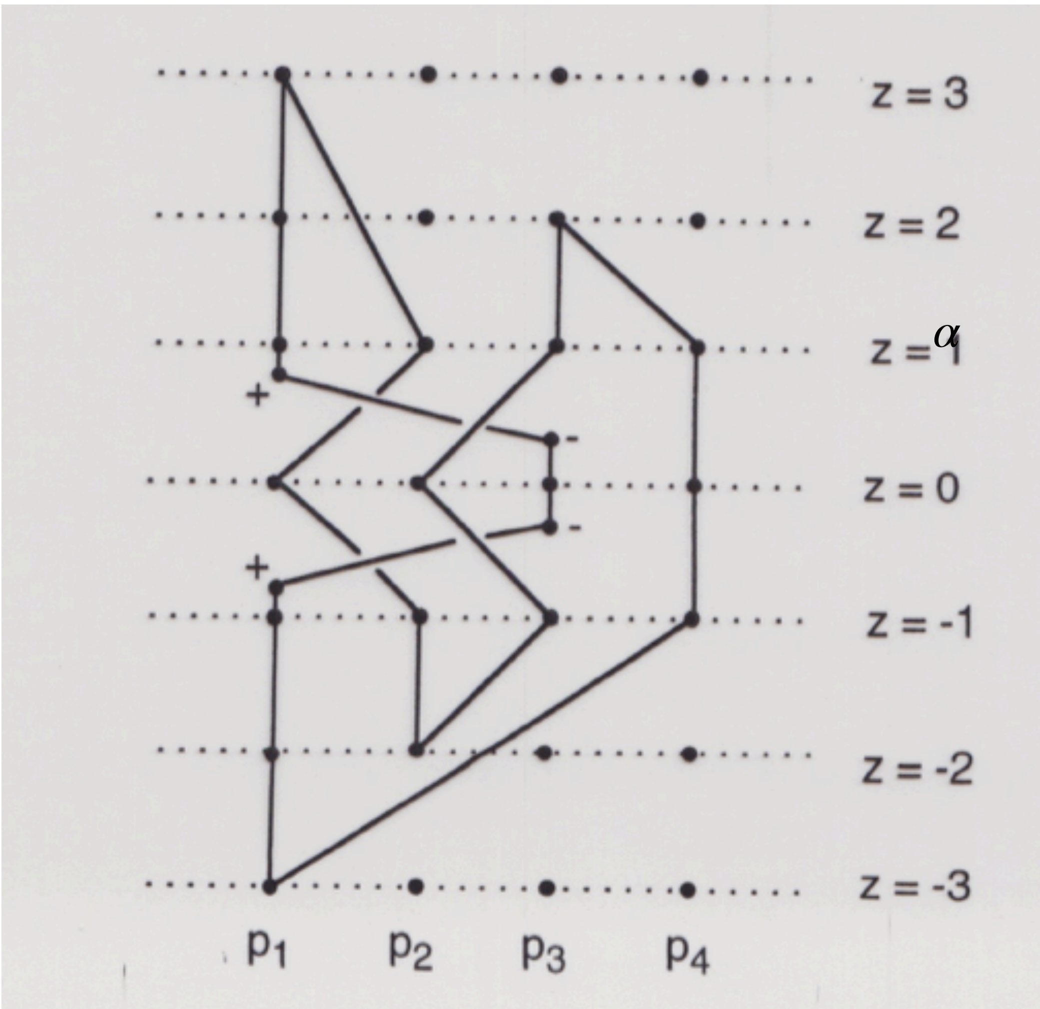


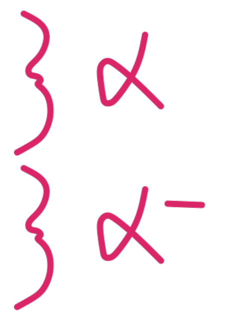
1. Iterating n times
2. Iterating n times
3. Capping off at the top and bottom

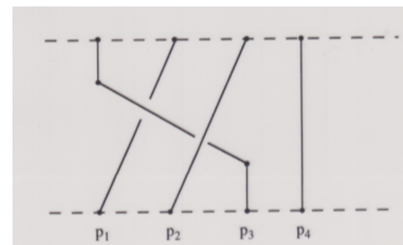


K_n is obtained by

1. Iterating α n times
2. Iterating α^{-1} n times
3. Capping off at the top and bottom

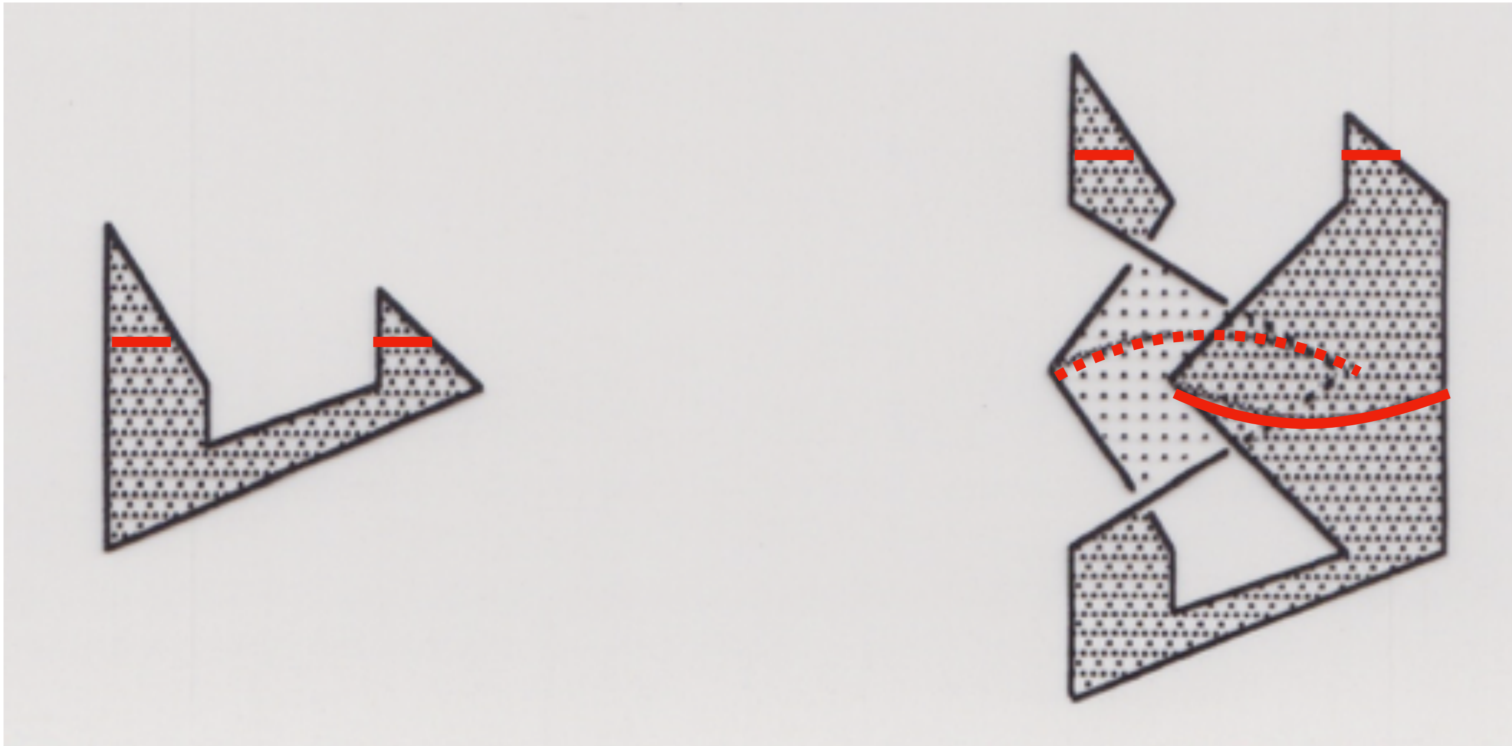






$$\alpha = \sigma_1 \sigma_2^{-1}$$

Spanning Disks for K_n

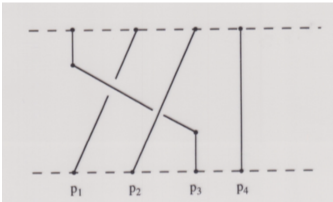


Standard disks with boundary K_n

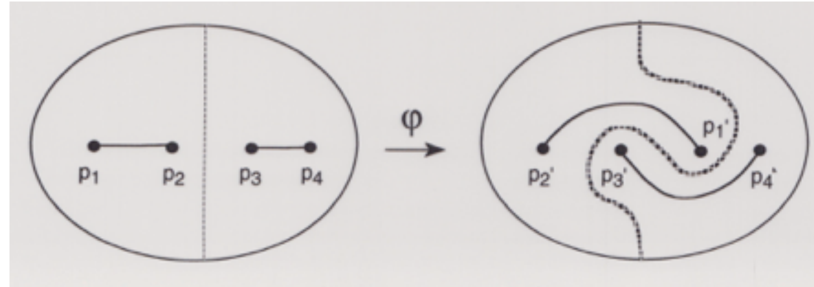
Each K_n is the boundary of a *standard* embedded disk in R^3.
We will show that

1. This disk cannot be constructed with less than 2^n flat triangles.
2. No other disk can do better.

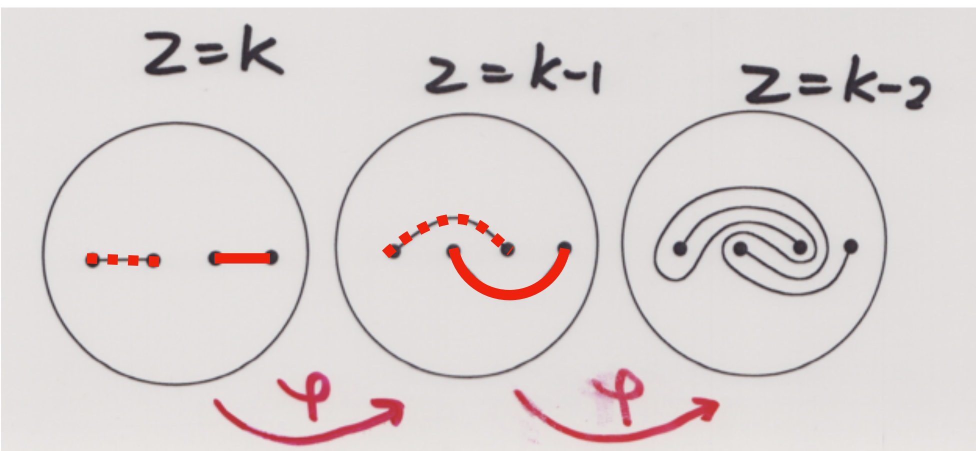
Braids and surface diffeomorphisms

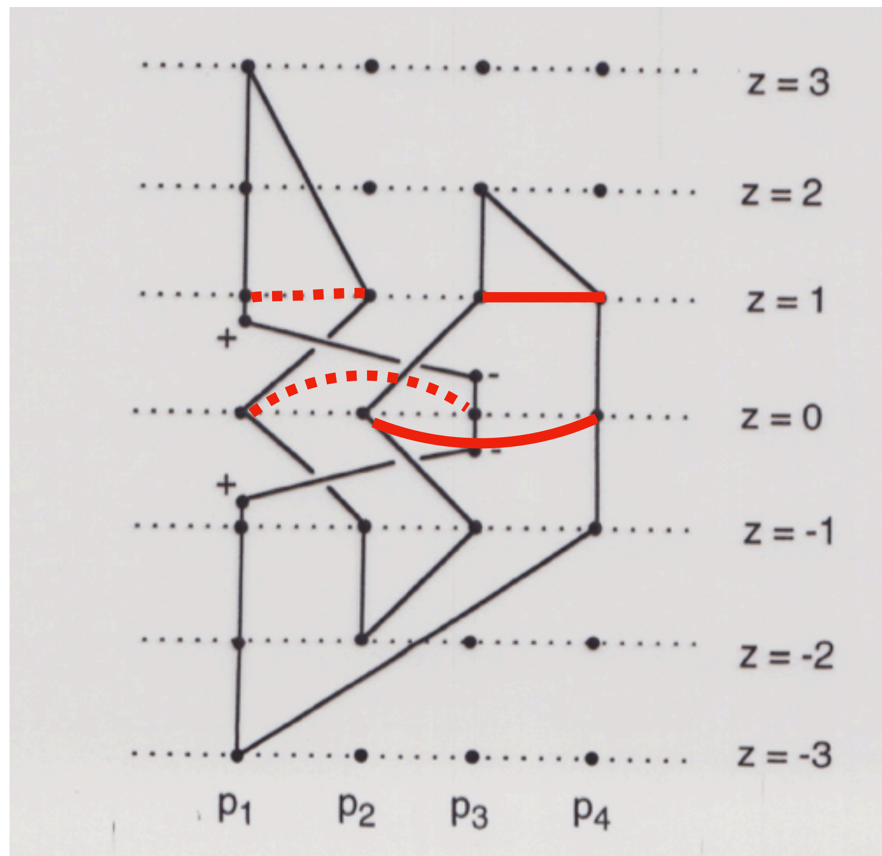


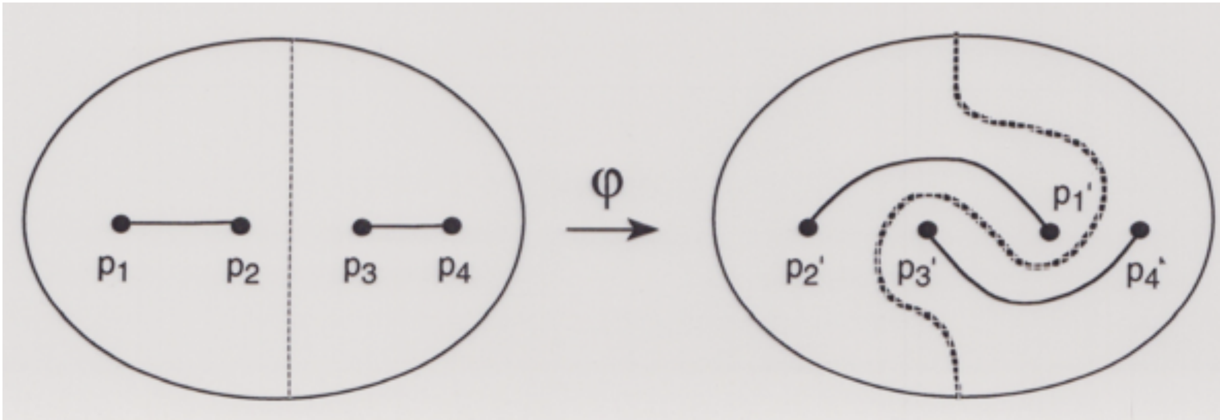
$$\alpha = \sigma_1 \sigma_2^{-1}$$



Associated to a braid is a diffeomorphism of a punctured disk.

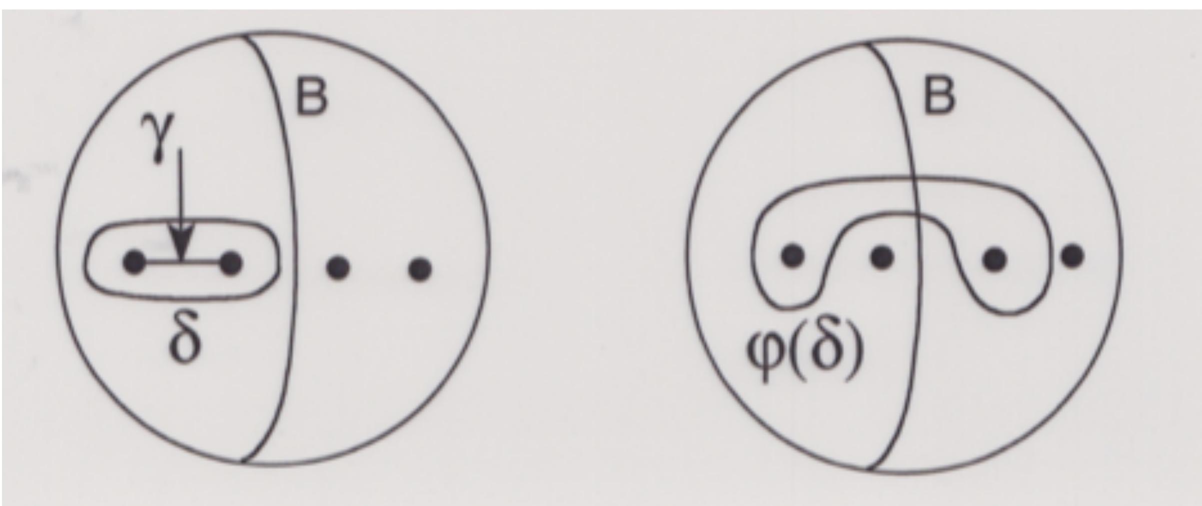


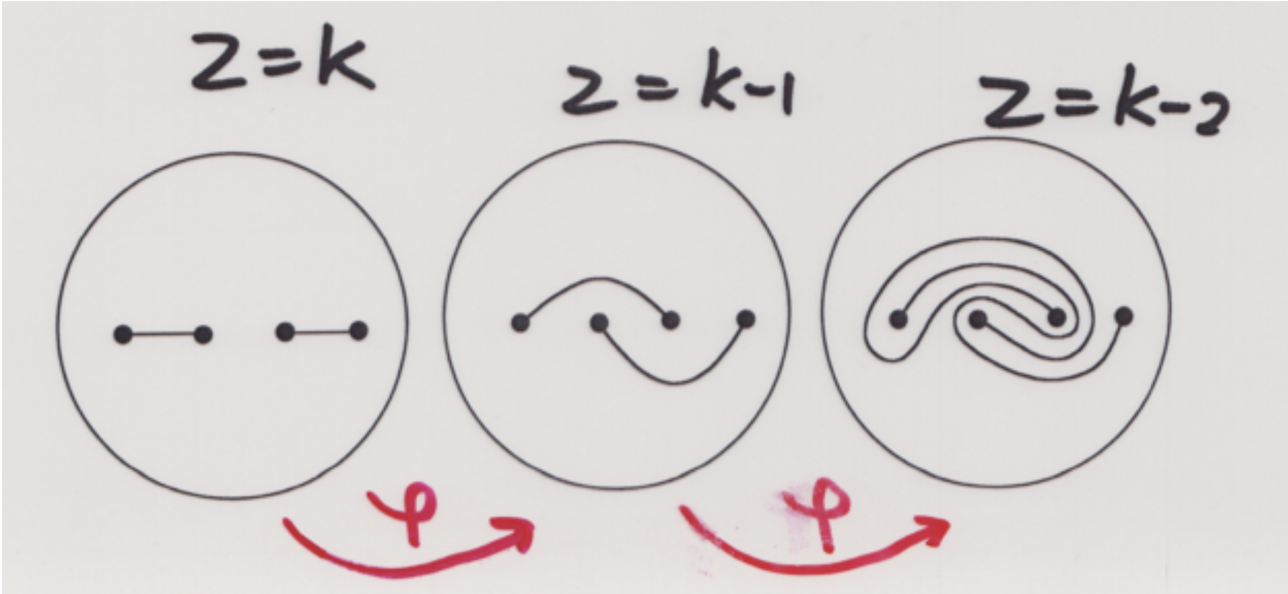


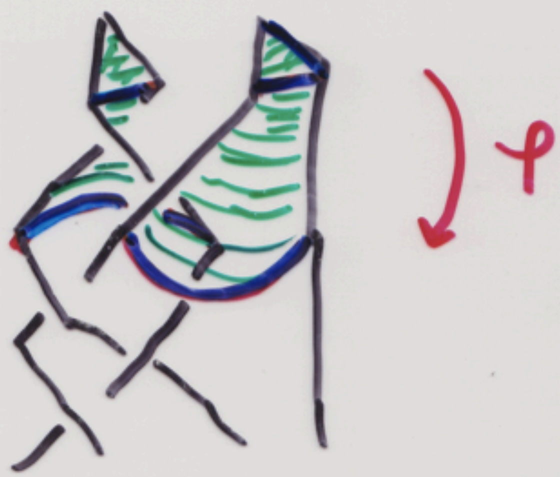


How can we understand the long term behavior of the sequence

$$\phi, \phi^2, \ldots$$



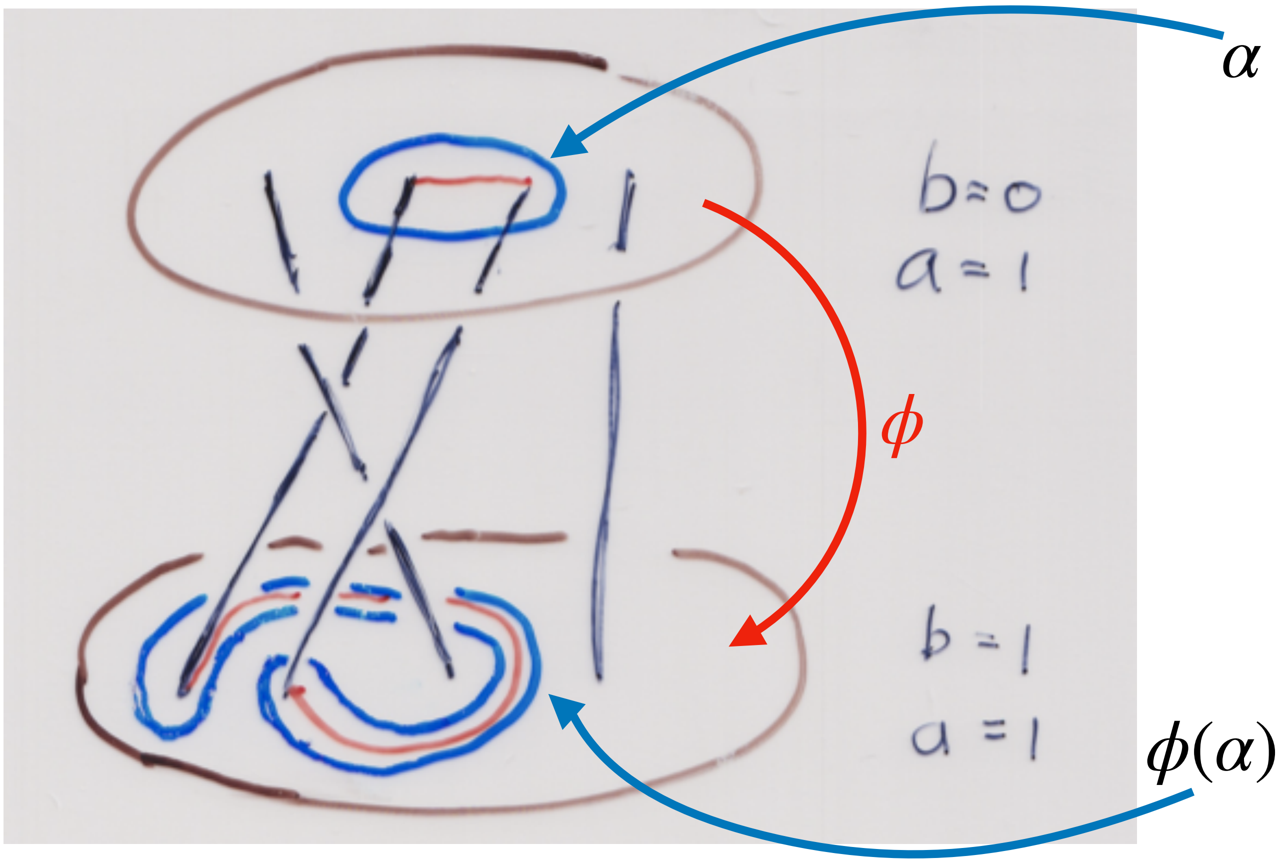




The level sets of a "standard" disk are iterated by φ.

$\varphi: S^2-(4 pts) \longrightarrow S^2-\{4 pts\}$

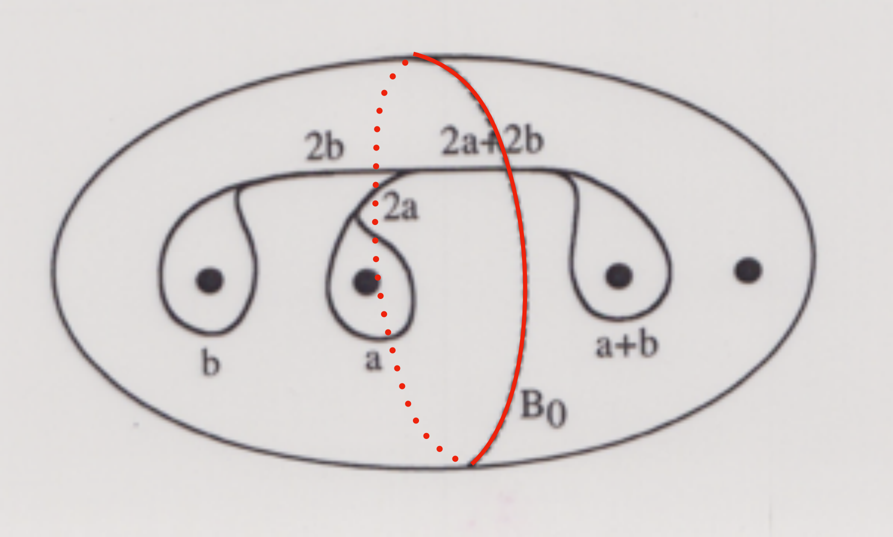
The level sets of a standard disk are stretched around the braid as we descend each level corresponding to an iterate of ϕ

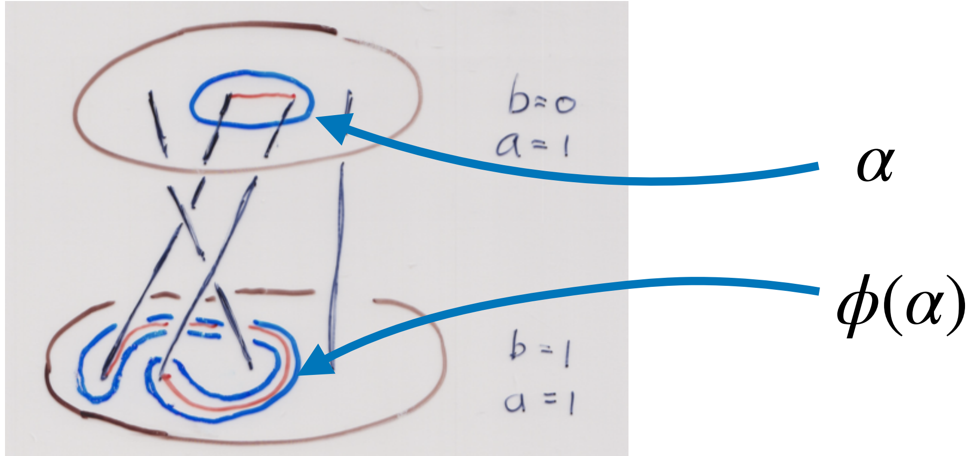


We are interested in the iterates of the loop α

To keep track of these, we use the theory of train tracks.

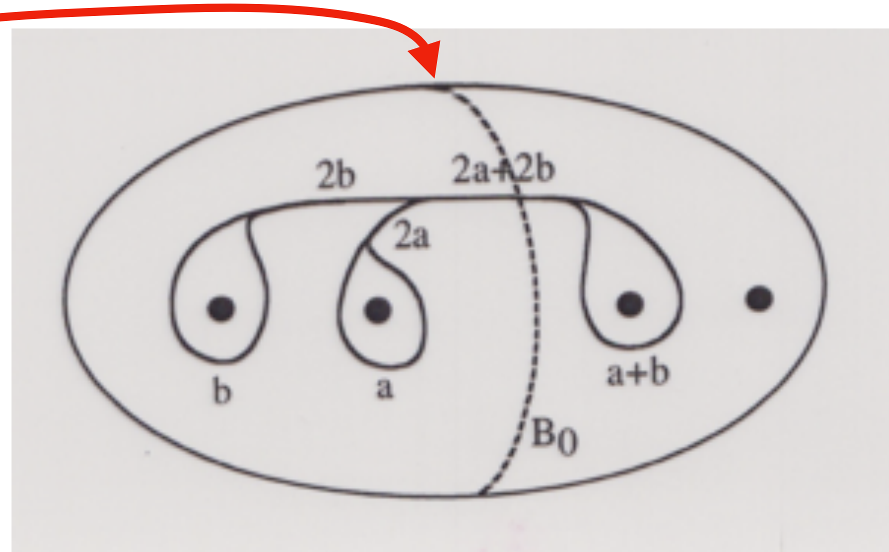
Each branch of a train track comes with weights. Fixing weights that add up appropriately at branches specifies a curve.



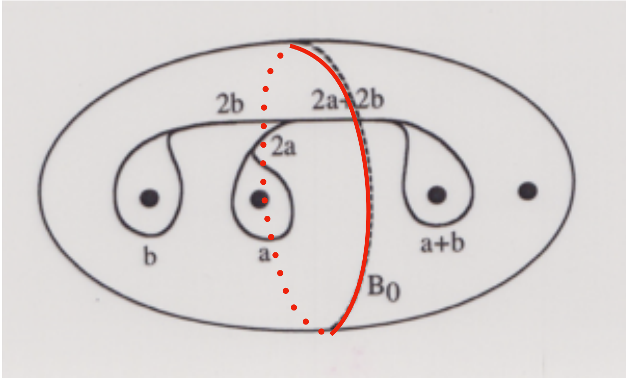


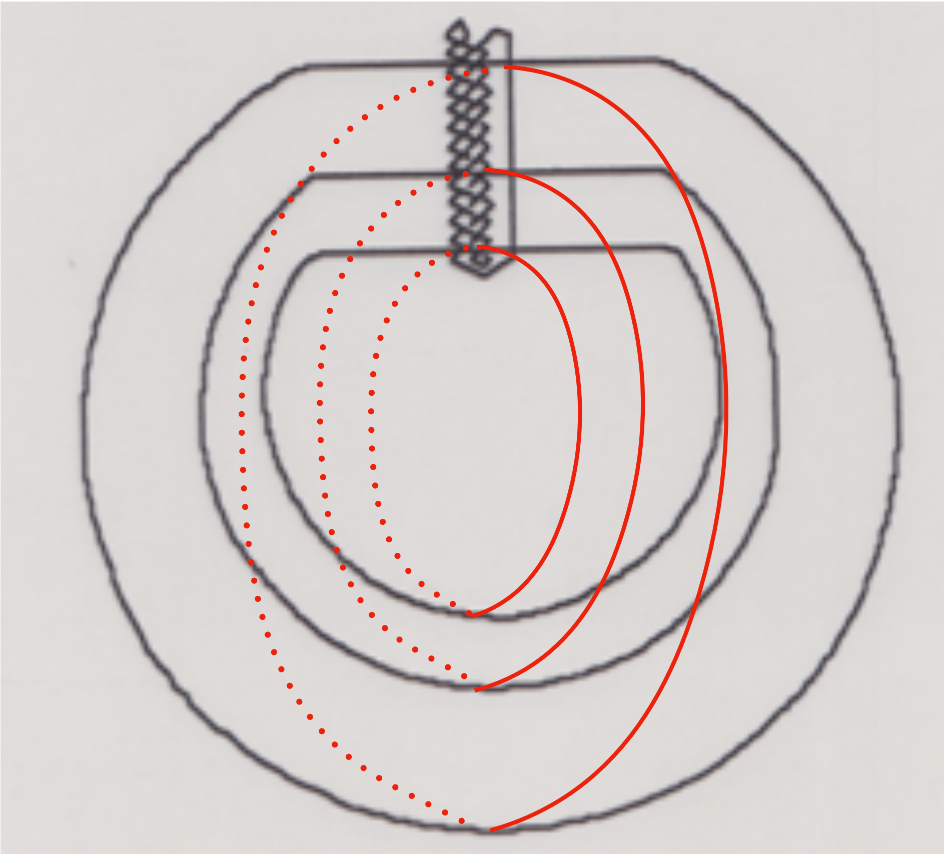
Train tracks give a way to understand the images of curves under iterated surface diffeomorphisms.
In our case we want to study the image of the blue loop α $(a = 1, b = 0)$ and its iterates $\phi^n(\alpha)$.

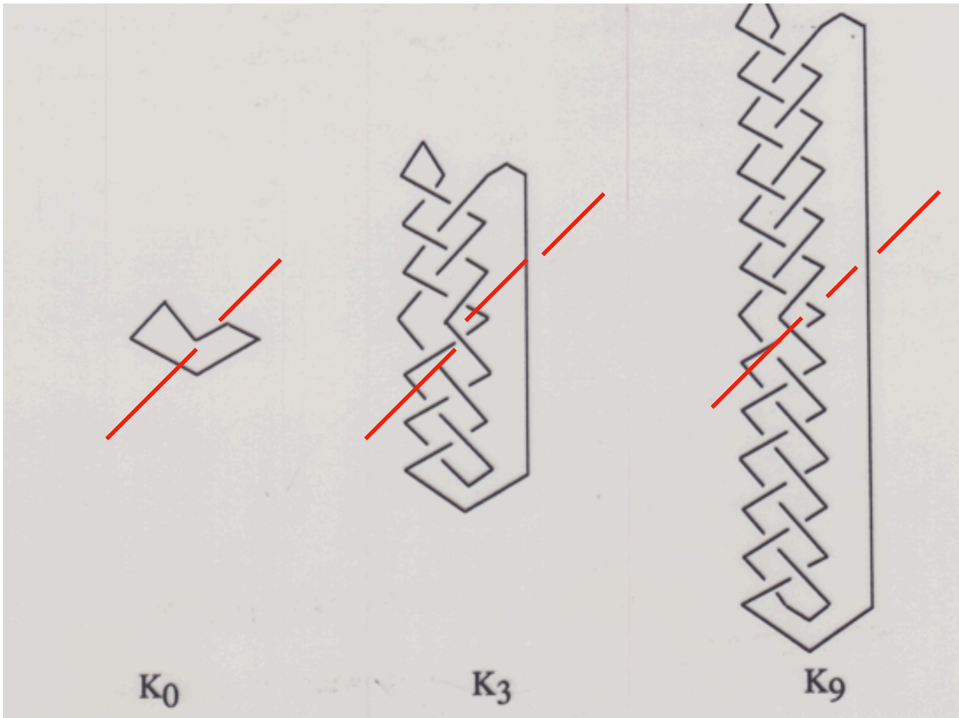
In particular we want to understand how the iterates $\phi^n(\alpha)$ intersect B_0.



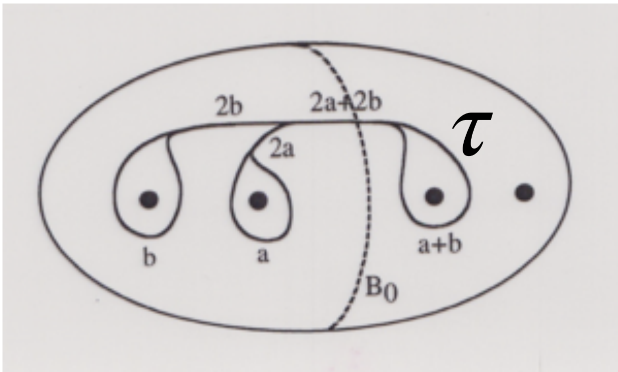
We want to understand how the iterates $\phi^n(\alpha)$ intersect B_0.







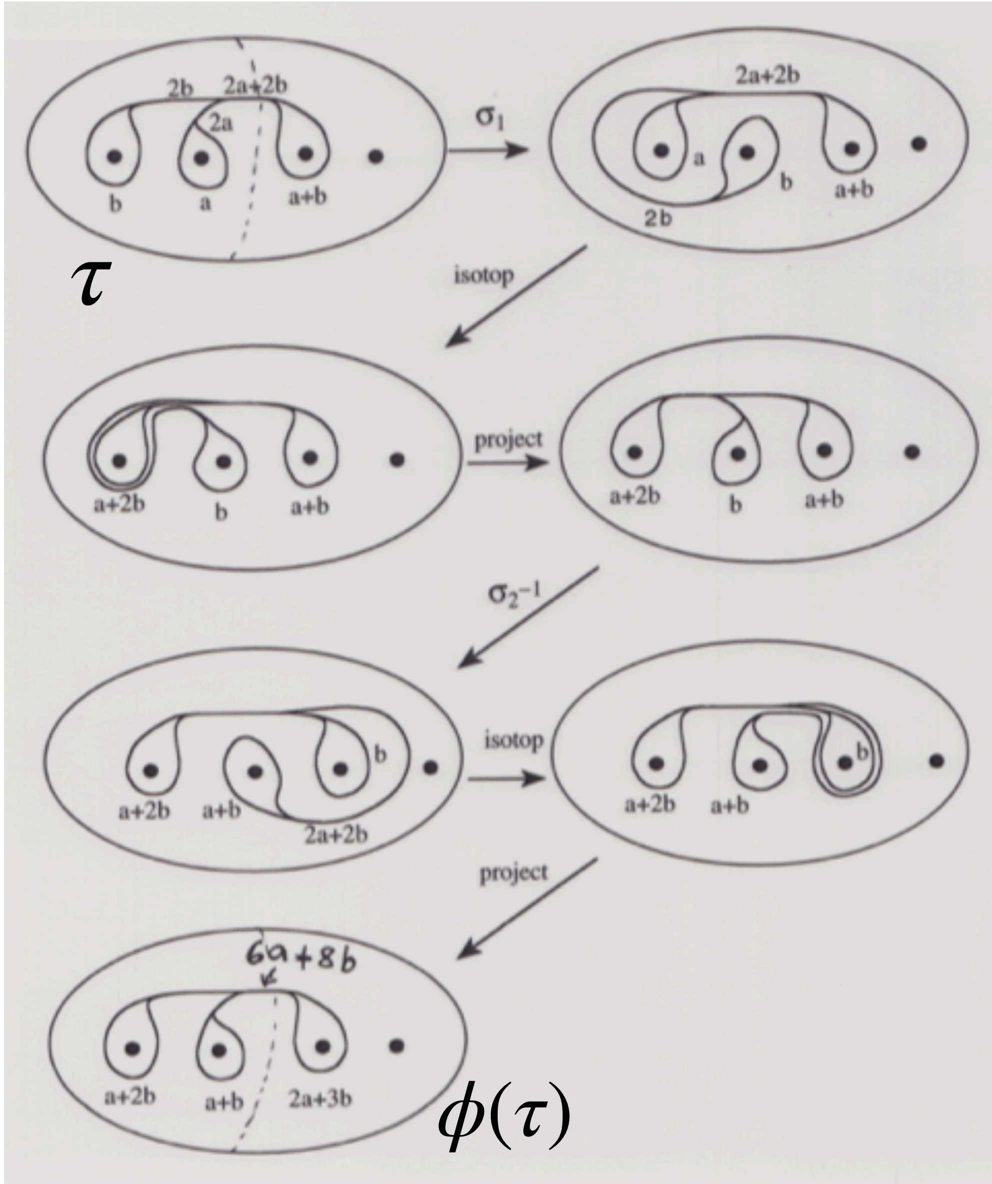
A train track *carries* a curve if some choice of weights gives that curve.



This train track τ carries the curve below with weights a=1, b=1. The other weights are determined.

Claim: This train track τ is taken to itself by ϕ . It is *invariant* under ϕ . Any curve carried by τ is also carried by $\phi(\tau)$.

Proof

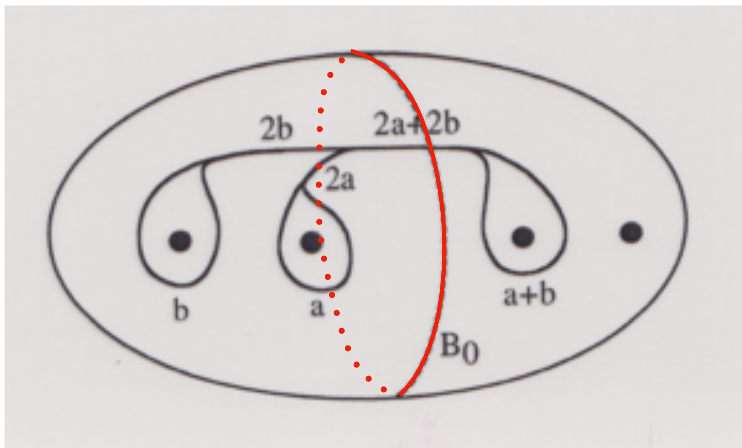


Each iteration of ϕ more than doubles the number of times that the standard disk spanning the curve intersects B_0.

$$2a + 2b \longrightarrow 6a + 8b \;>\; 2\,(2a + 2b)$$

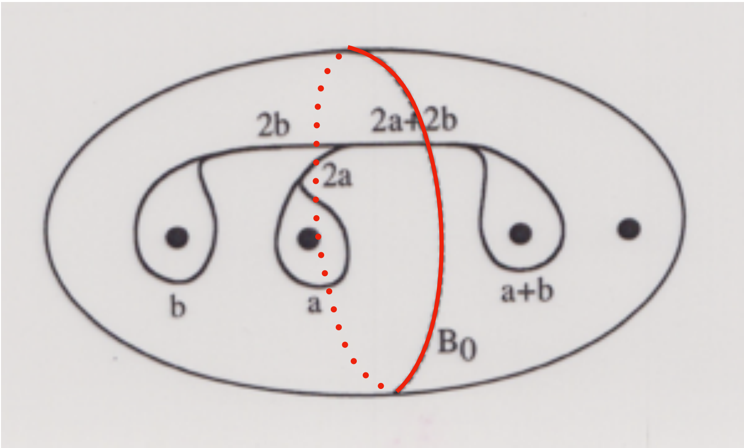
Each iteration of φ more than doubles the number of times that a standard disk spanning the curve intersects B_0.

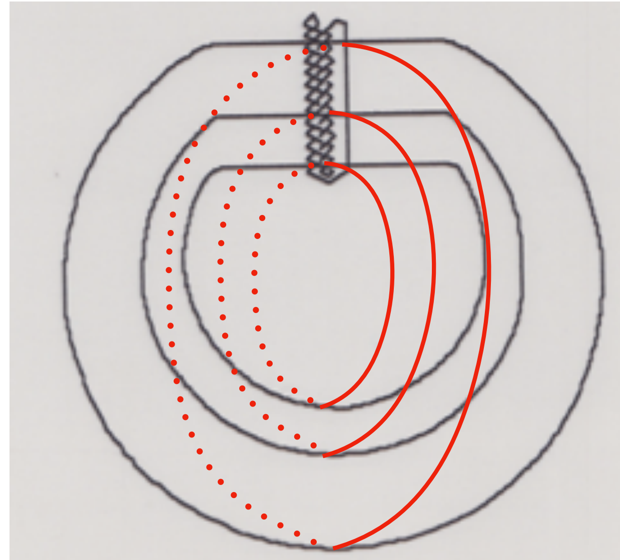
$2a + 2b \longrightarrow 6a + 8b > 2\,(2a + 2b)$



The number of times a standard disk intersects B_0 more than doubles under each iteration of ϕ . B_0 is a straight line when it intersects $\phi^n(\alpha)$ in the level set at height n.

If the standard disk is triangulated, it must have at least 2^n triangles, since each triangle intersects a line at most once.





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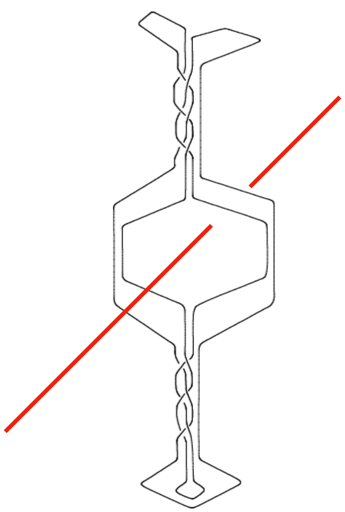
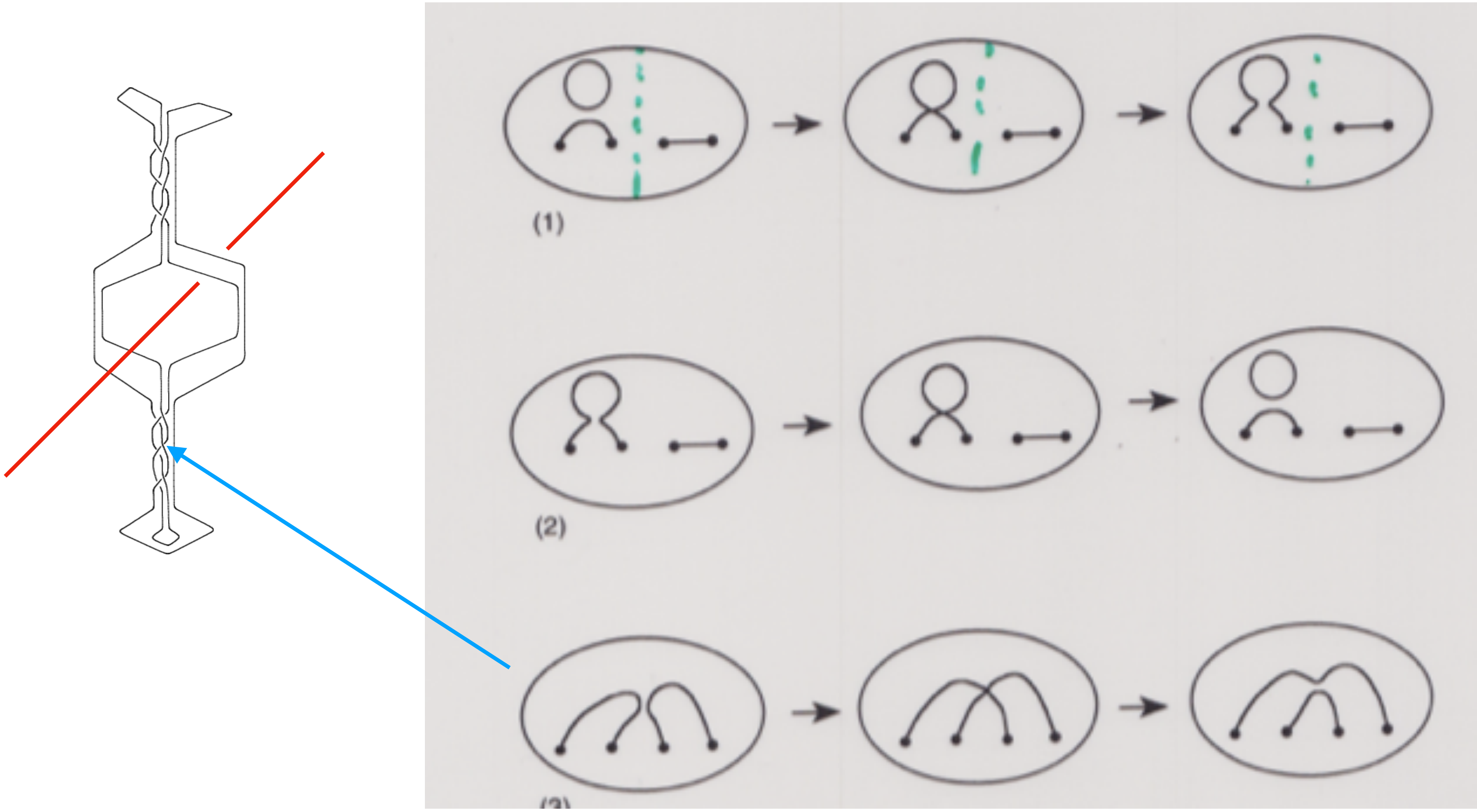


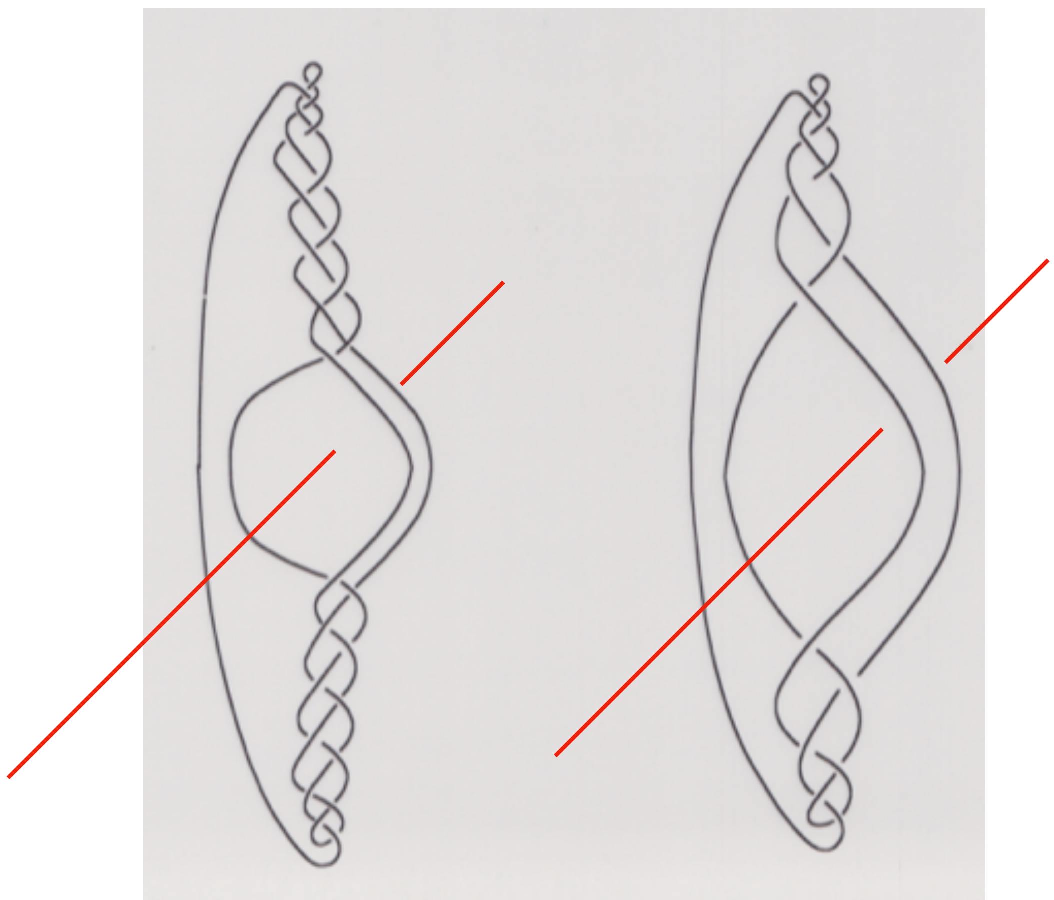
What if we looked at some other disk spanning K_n, rather than the standard disk. Could it intersect B_0 in less points?



Look at the level sets of a Morse function for any disk spanning K_n. Type 1 and 2 critical points don't affect the number of intersections with B_0. Type 3 do change this number, perhaps drastically. But only one type 3 can occur. So the argument applies at the middle level, either working up from the bottom, or down from the top.

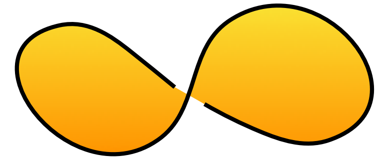
Conclude:

Any disk with boundary K_n must have at least 2^n triangles, since such a disk must cross the red line at least 2^n times, and each triangle intersects a line at most once.



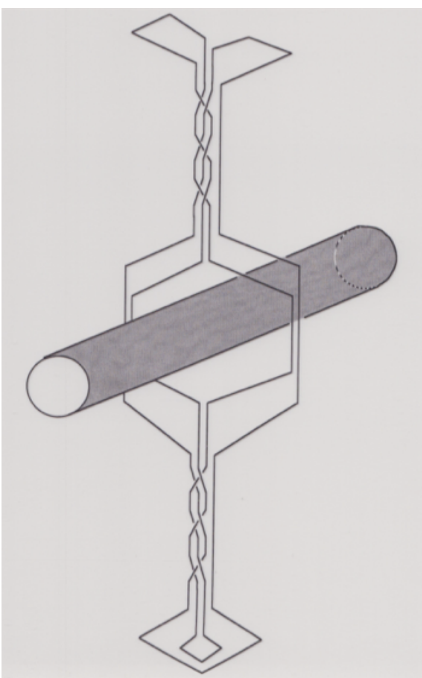
Normal and minimal surfaces

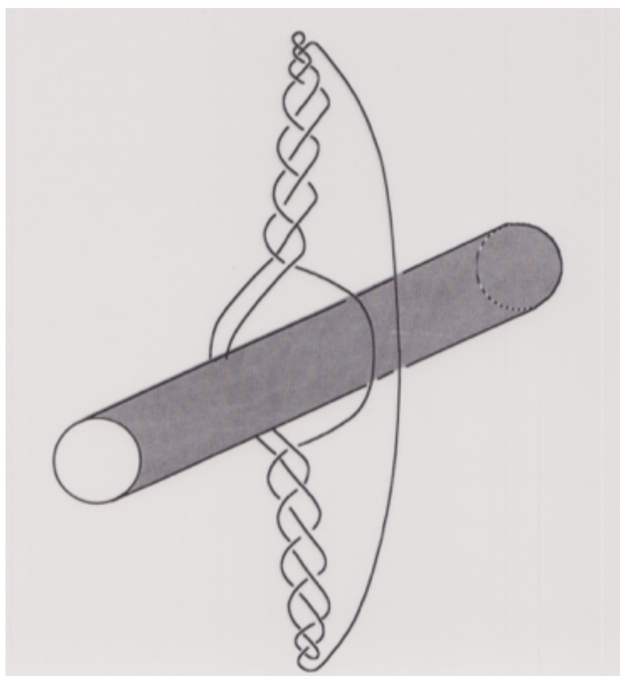
What bounds can we get on an embedded disk?



Theorem 1. (H-Lagarias-Thurston, 2004)
There is a sequence of unknotted, smooth curves γ_n embedded in R^3, each having length L = 1, such that the area of any embedded disk spanning γ_n is greater than n.

Theorem 1 holds for the curves below if they are normalized to have length one. Any disk spanning these curves crosses the cylinder below exponentially often.

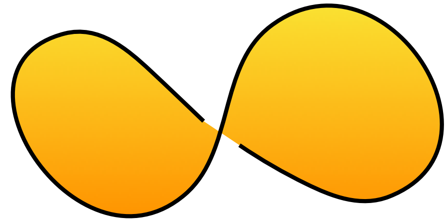




Question.
Can we control the area of a spanning disk by adding some additional geometric condition?

Normal and minimal surfaces

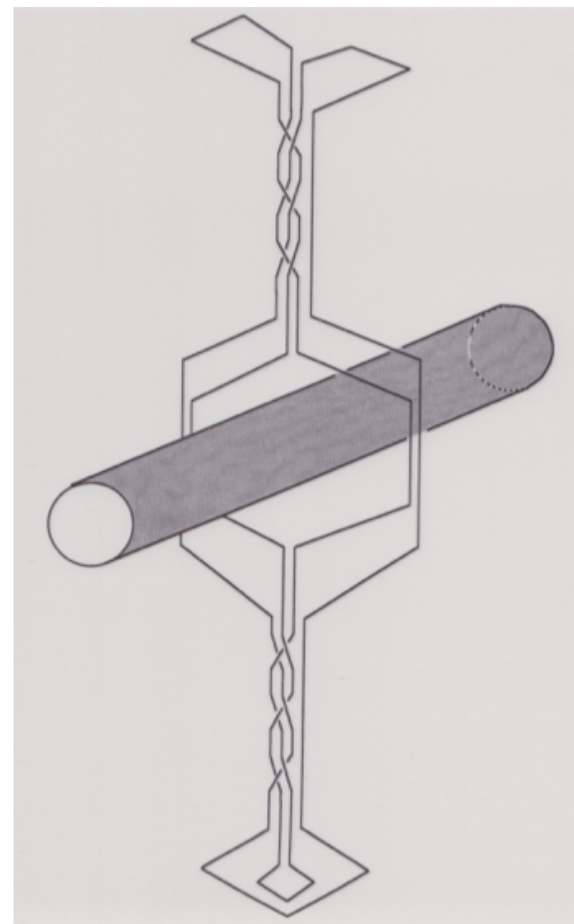
What bounds can we get on an embedded disk?



Theorem 2.
For any embedded closed unknotted smooth curve γ in R^3 having length L and thickness r, there exists a smooth embedded disk of area A, having γ as boundary with

$$A \leq (C_0)^{(L/r)^2} L^2$$

where $C_0 > 1$ is a constant independent of γ, L and r.



For a curve with length one:

$$A \leq (C_0)^{(1/r)^2}$$

Normal and minimal surfaces

Theorem 2.
For any embedded closed unknotted smooth curve γ in R^3 having length one and thickness r, there exists a smooth embedded disk of area A, having γ as boundary with

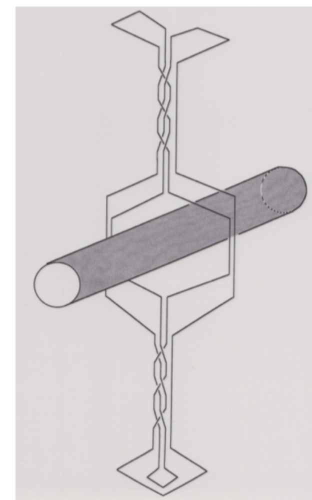
$$A \le (C_0)^{(1/r)^2}$$

Proof. Isotop γ within its (1/r) tubular neighborhood to a polygon K with n edges, where

$$n \le 32(1/r)$$

Triangulate the complement of K in a ball B of radius 4. B contains less than t tetrahedra by an explicit construction, where

$$t = 290n^2 + 290n + 116$$

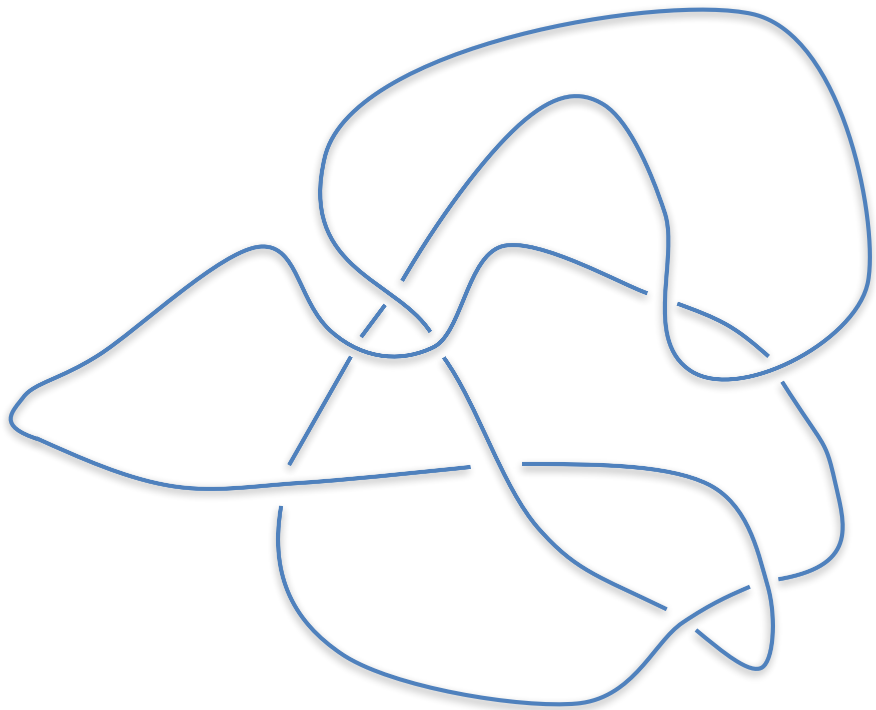


Then construct a spanning disk for γ that is a *fundamental normal disk*. This requires at most C_2 disks, where

$$C_2 = 2^{10^8 t}$$

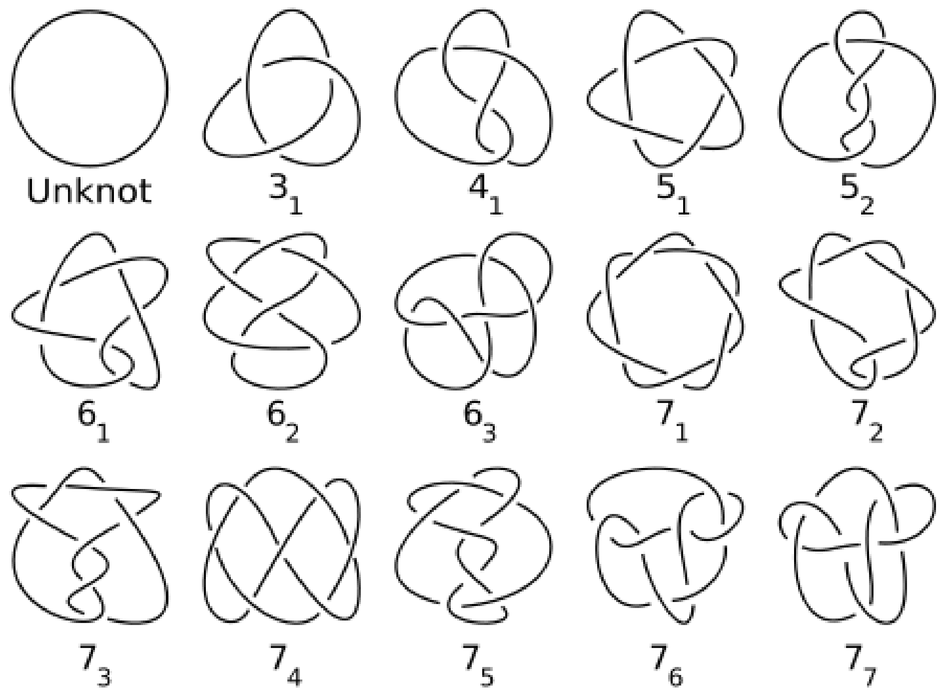
Each disk is a triangle in a ball of radius 2, and thus has area at most 8. Sum up the areas to get an upper bound.

Knot and Link Diagrams



The study of knot diagrams - planar curves with choices of over and under-crossings is an interesting subject of its own.

Knot diagrams



- Traditionally - diagrams are used to study knots and links.
- Diagrams are interesting in their own right.
- We can reverse the usual approach - use knots and links to study diagrams.
- The space of diagrams has more structure than the space of knots.

Lower Bounds for Reidemeister Moves

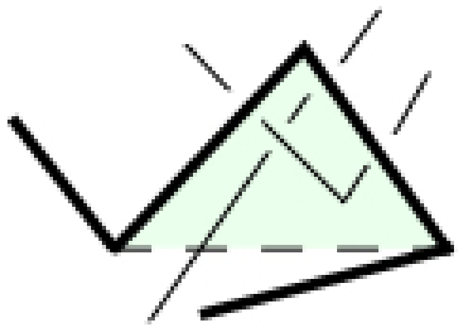
Suppose that any *n* crossing unknot diagram D_n can be transformed to the trivial diagram with *U(n)* Reidemeister crossings.

What do we know about *U(n)?*

Bounds for U(n)

Theorem H-Lagarias (2001): $U(n) \leq (2^{10})^n$
Idea: Compute an upper bound for the number of triangles in a Fundamental Normal Surfac
Surface. The Fundamental surfaces include an unknotting disk if K is the unknot.

Slide the knot across one triangle of this disk at a time.



Each move across a triangle can result in $(C_1)^t$ Reidemeister moves and there can be put such $(C_2)^t$ triangles to slide across. The resulting number of Reidemeister moves is bound by

$$(C_1)^t(C_2)^t = (C_3)^t$$

This method cannot improve the bound from exponential to polynomial.

Bounds for U(n)

Theorem Lackenby (2013)

$$U(n) \leq (231n)^{11}$$

Theorem H-Nowik (2010)

$$U(n) \geq \frac{n^2}{25}$$

Open Problem: Close the gap.
For example, find candidate examples requiring more than quadratic numbers of Reidemeister moves, to show that

$$U(n) \geq Cn^3$$

To Establish LOWER Bounds:

Use EXAMPLES and INVARIANTS

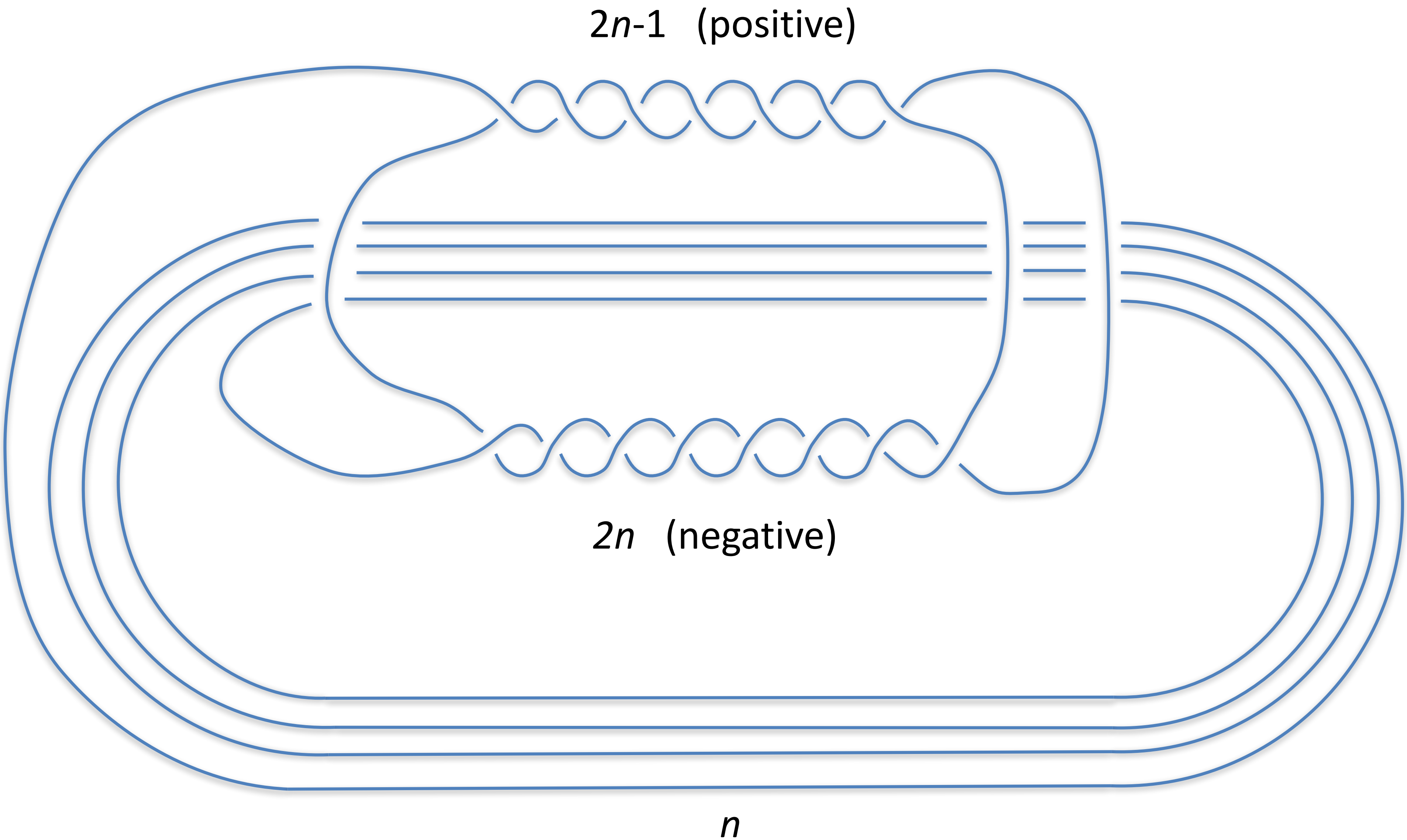
1. Find a family of unknot diagrams D_n that seem to require a lot of Reidemeister moves to simplify.

2. Show that they really do require a lot of moves by constructing and computing Diagram invariants.

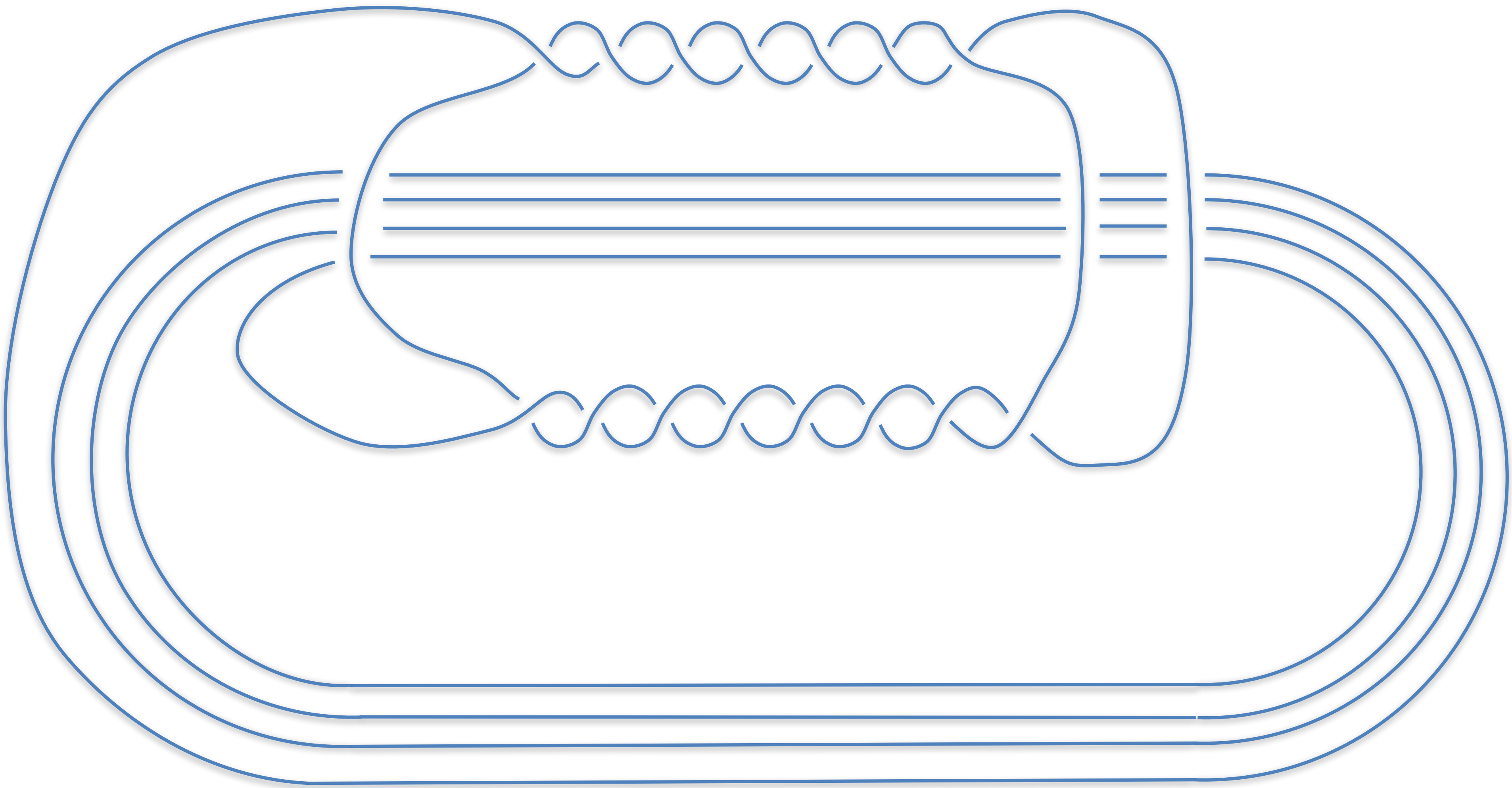
The Examples Giving the Best Known Lower Bounds

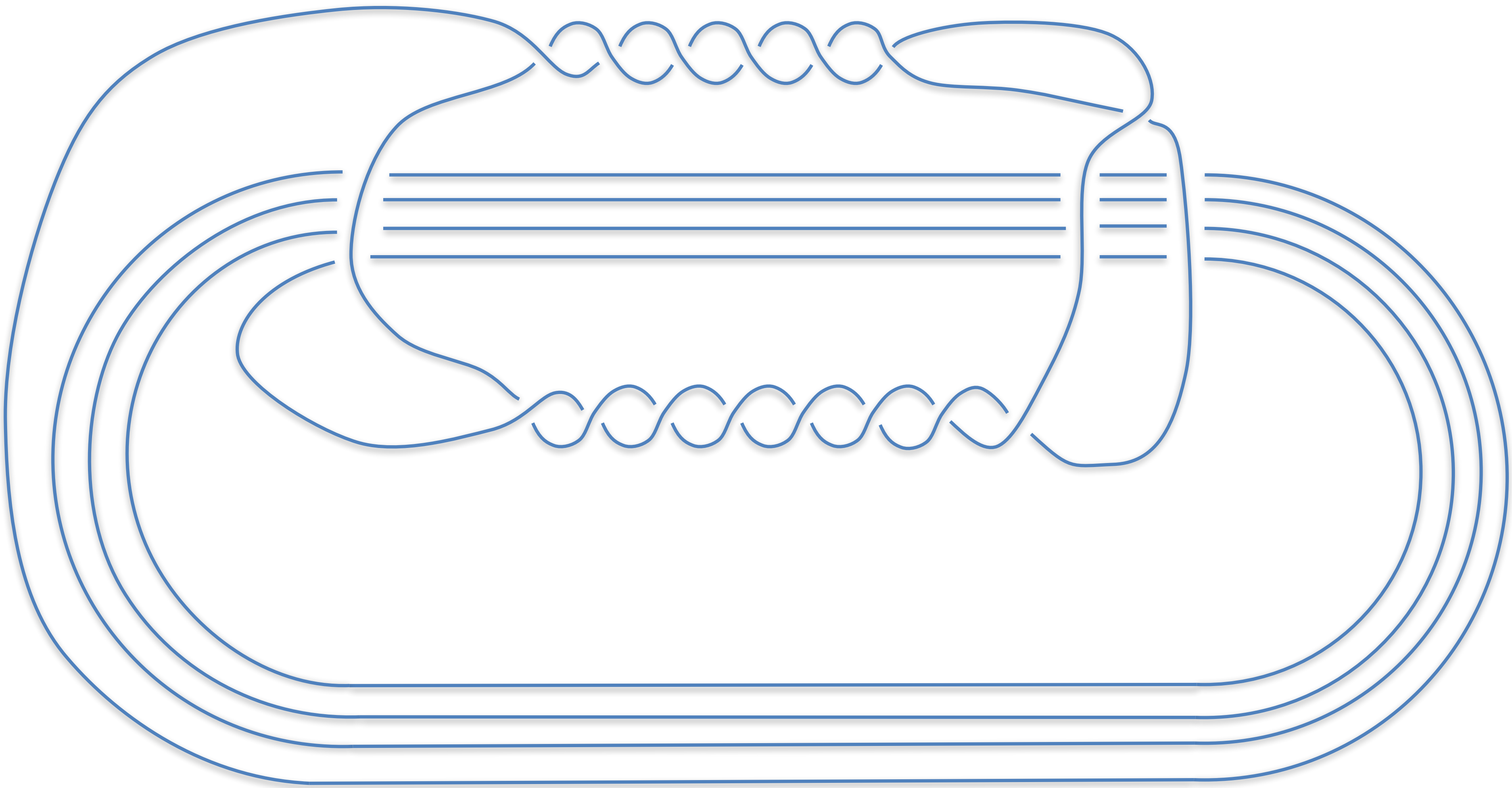
$D_n \quad (n = 4)$

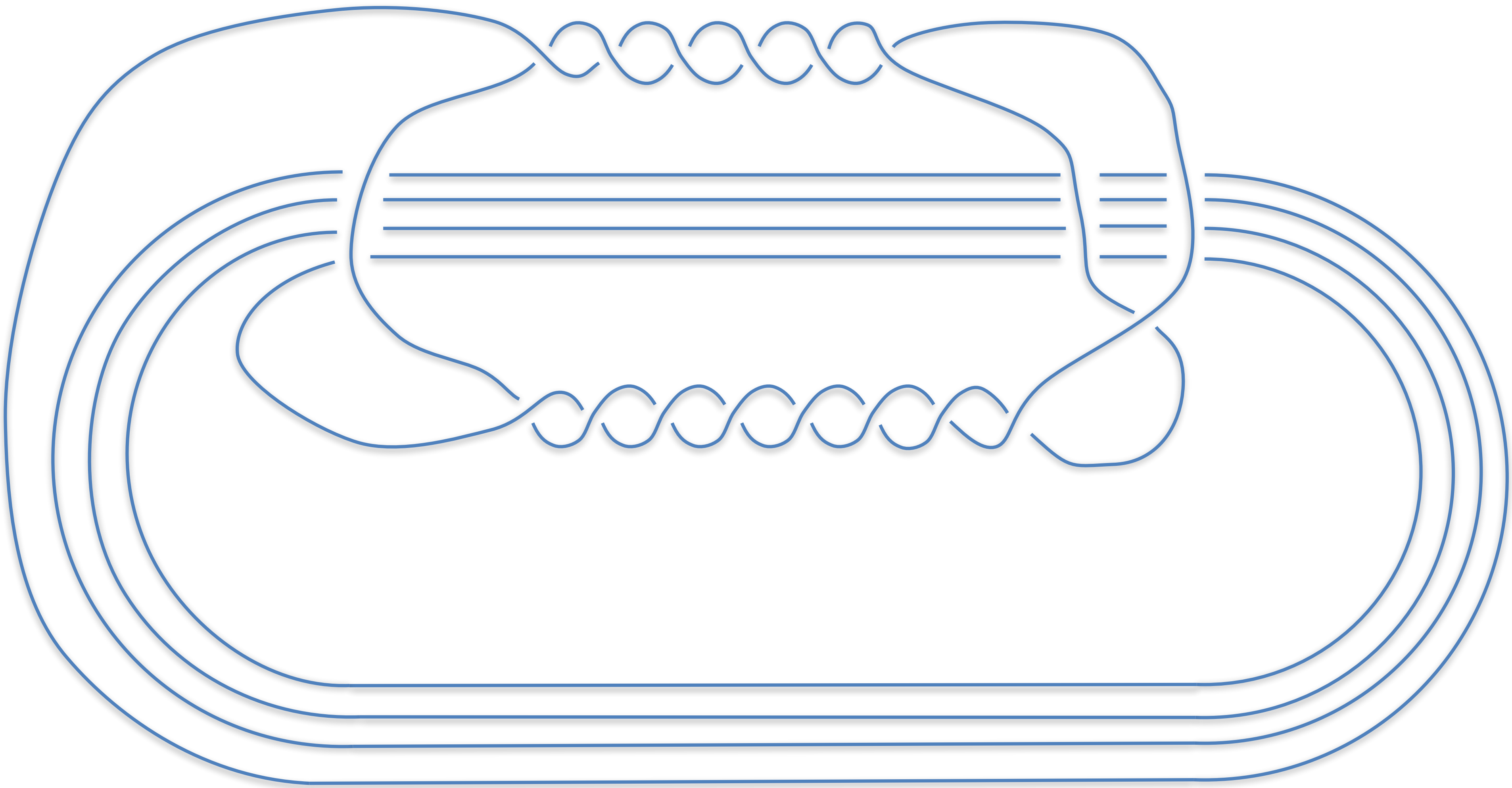
$7n - 1$ crossings

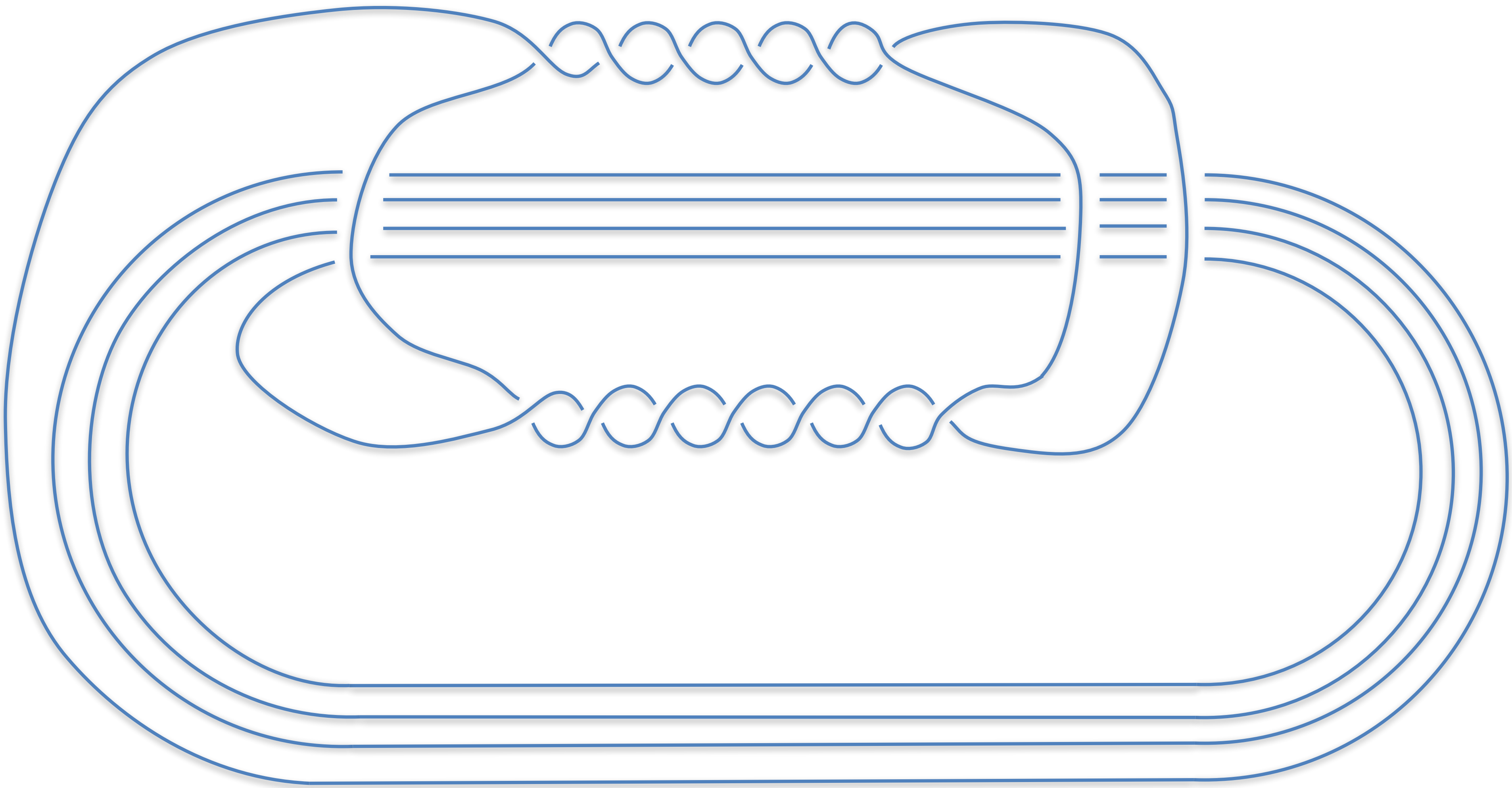


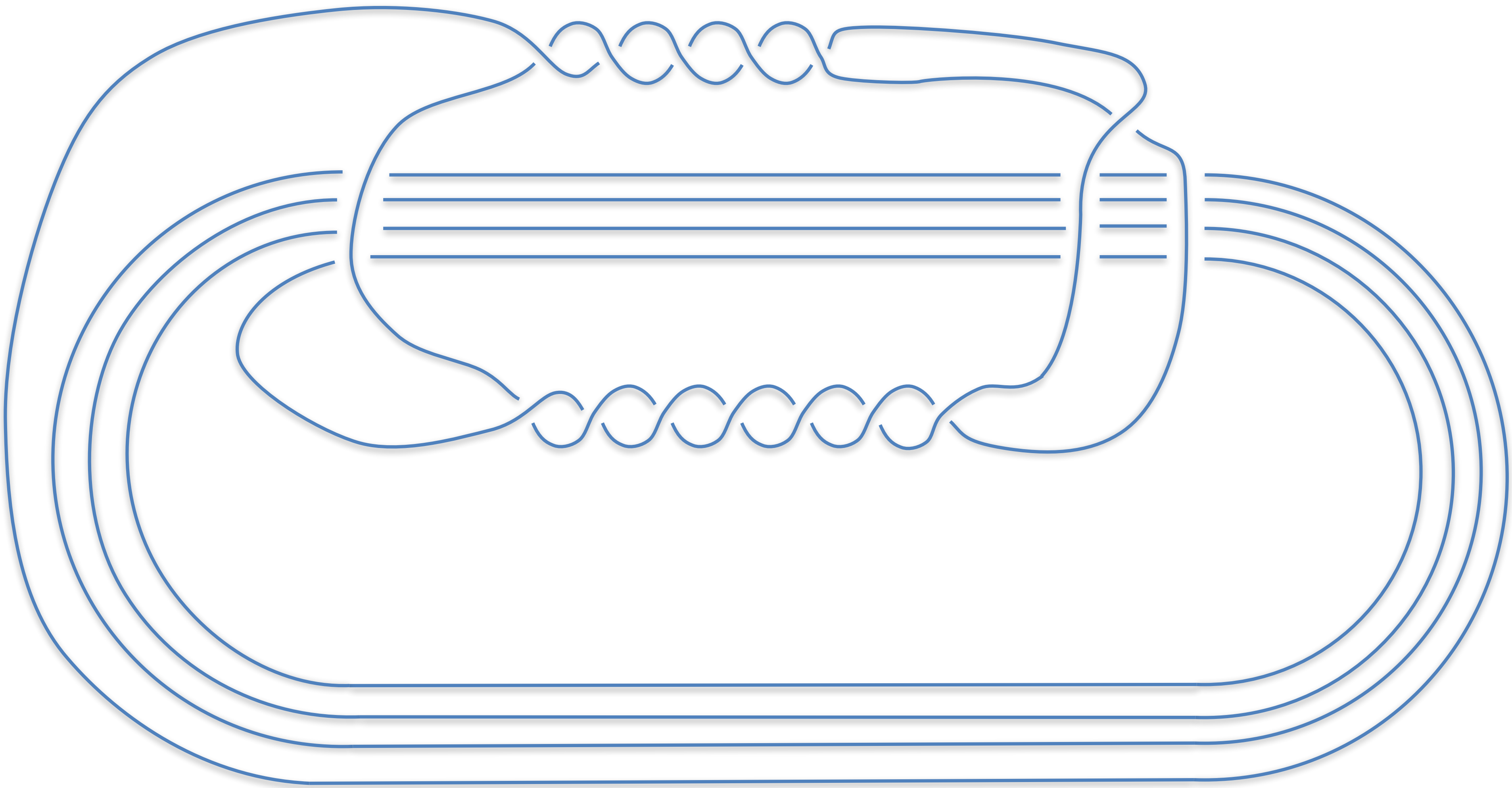
Previous lower bounds were linear.

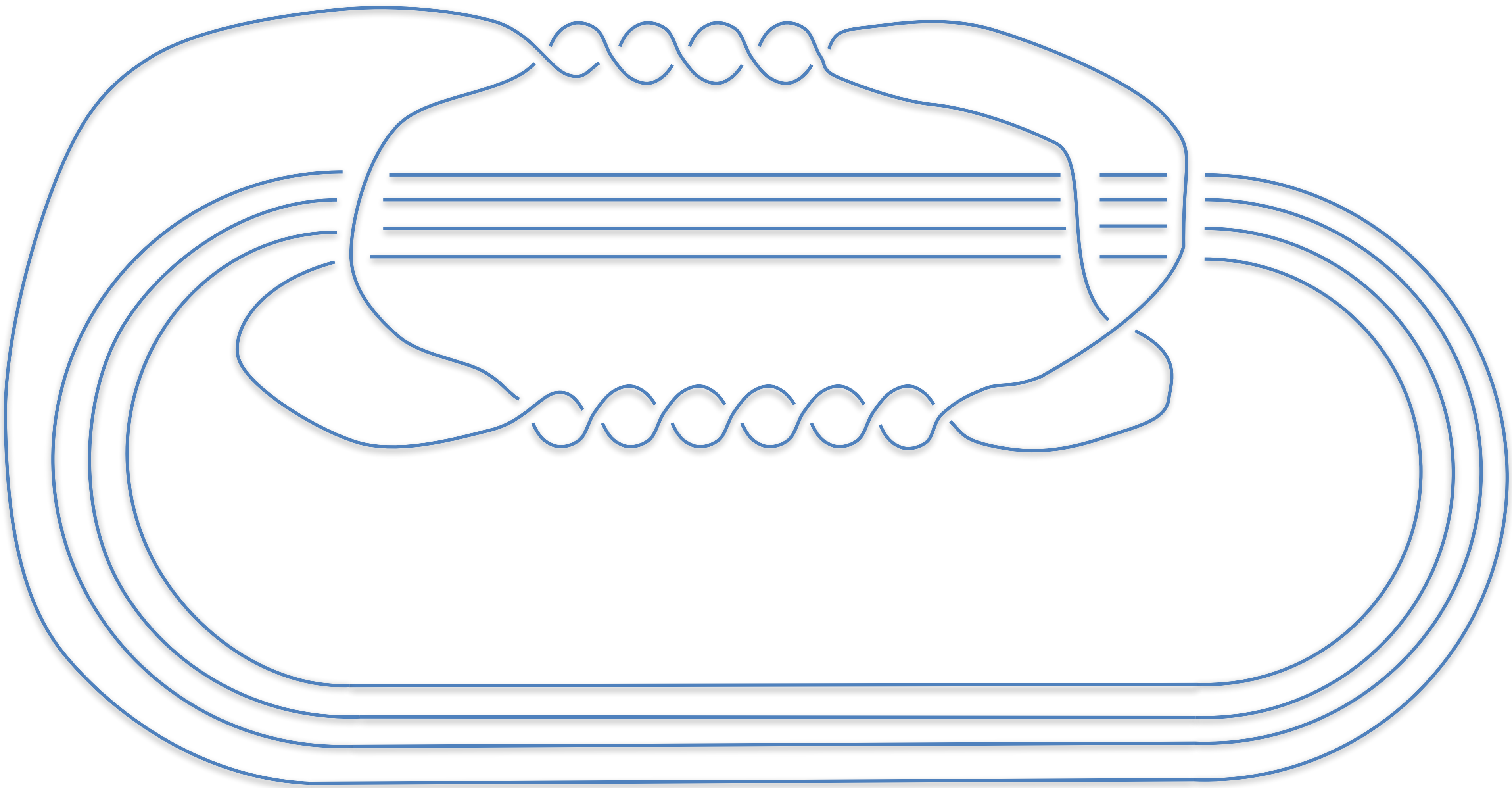


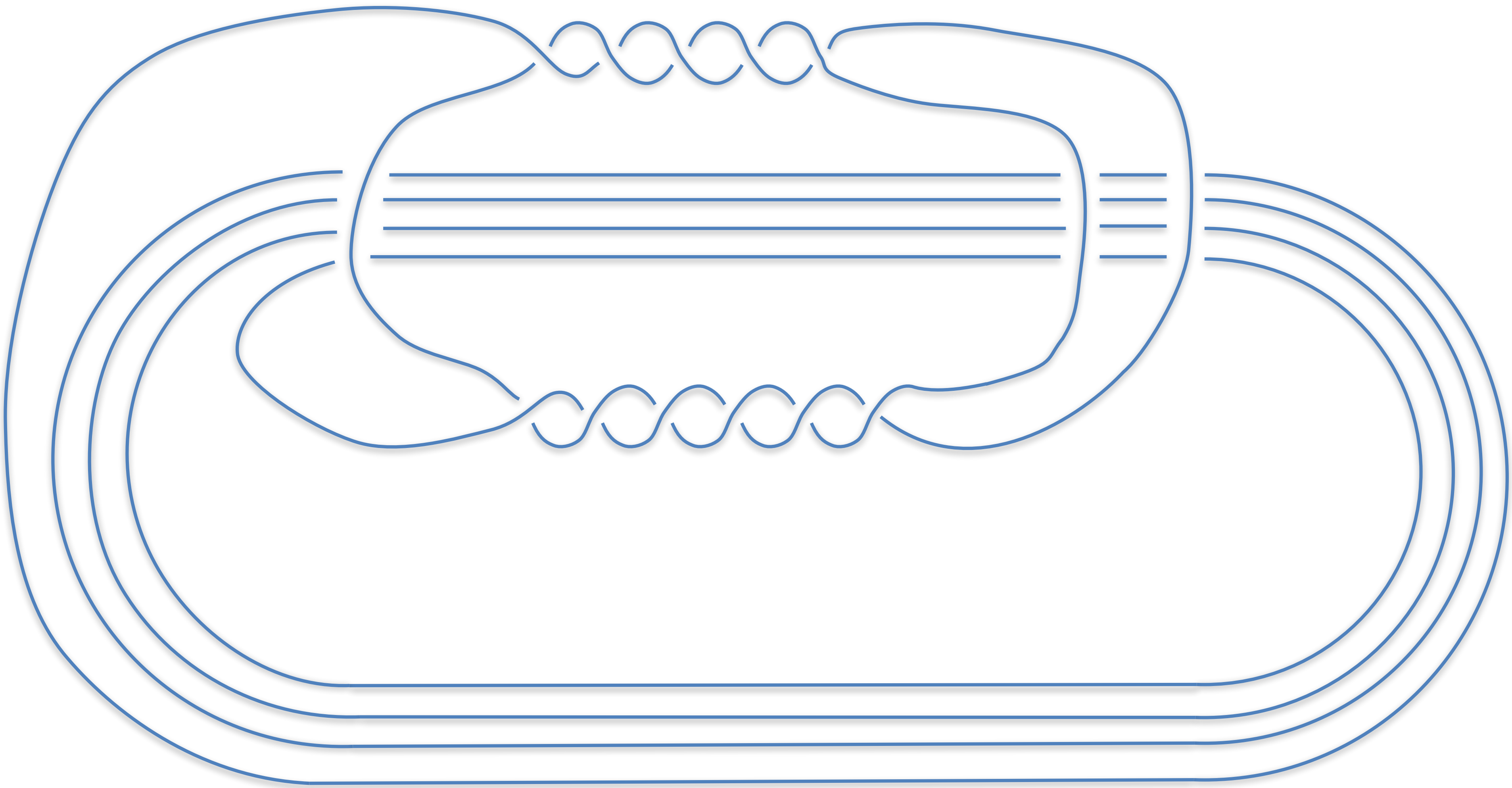


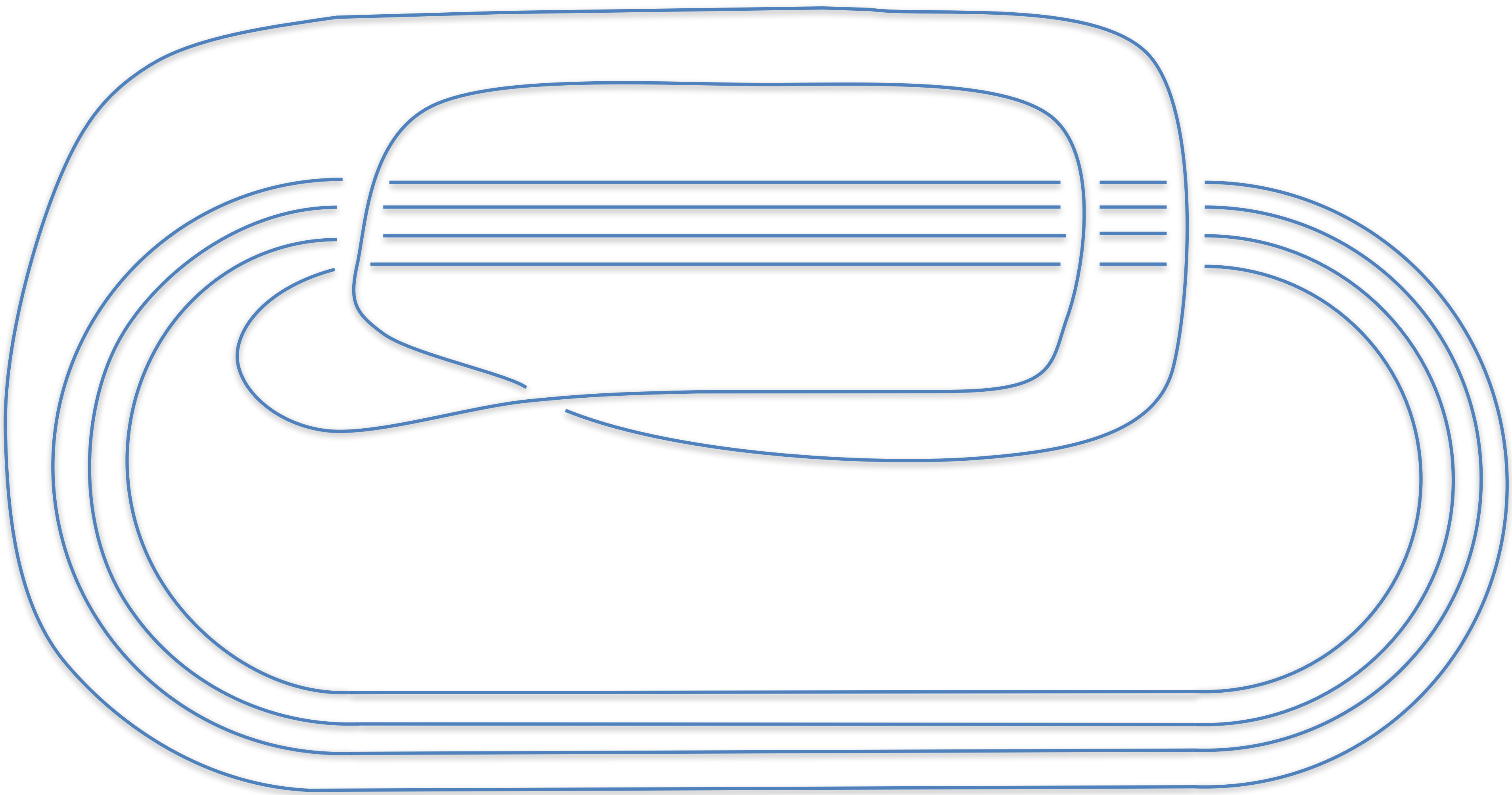


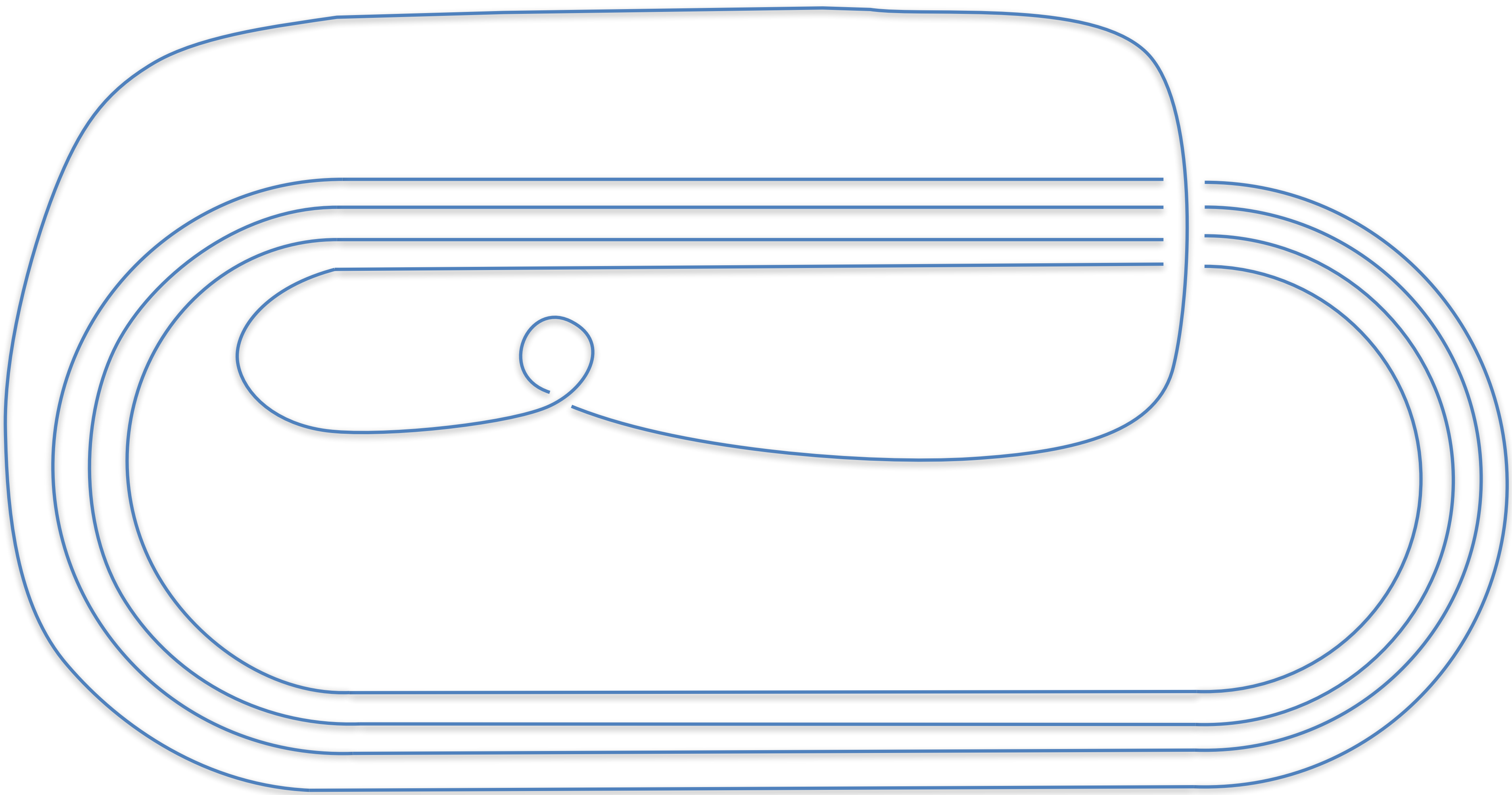


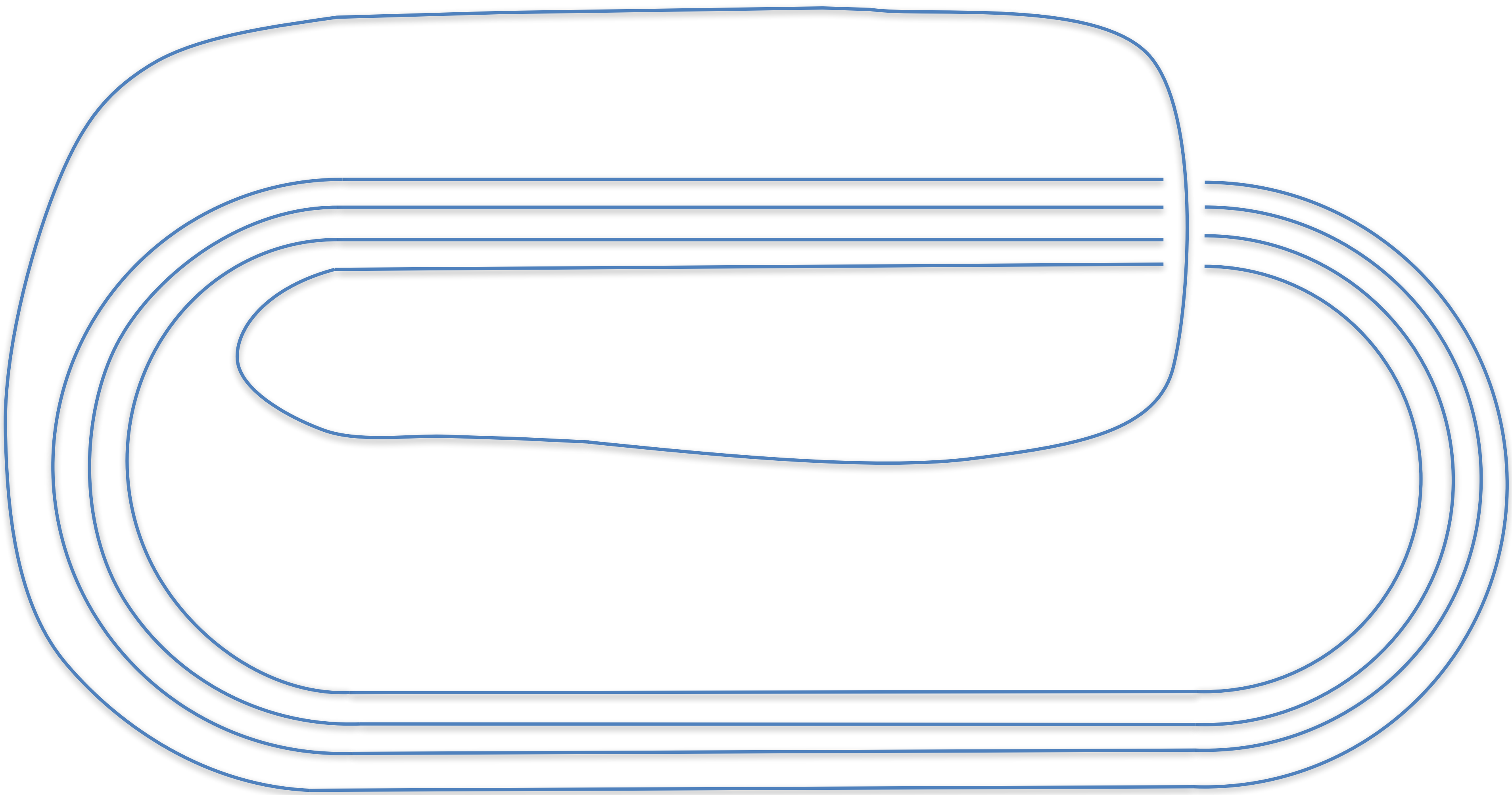


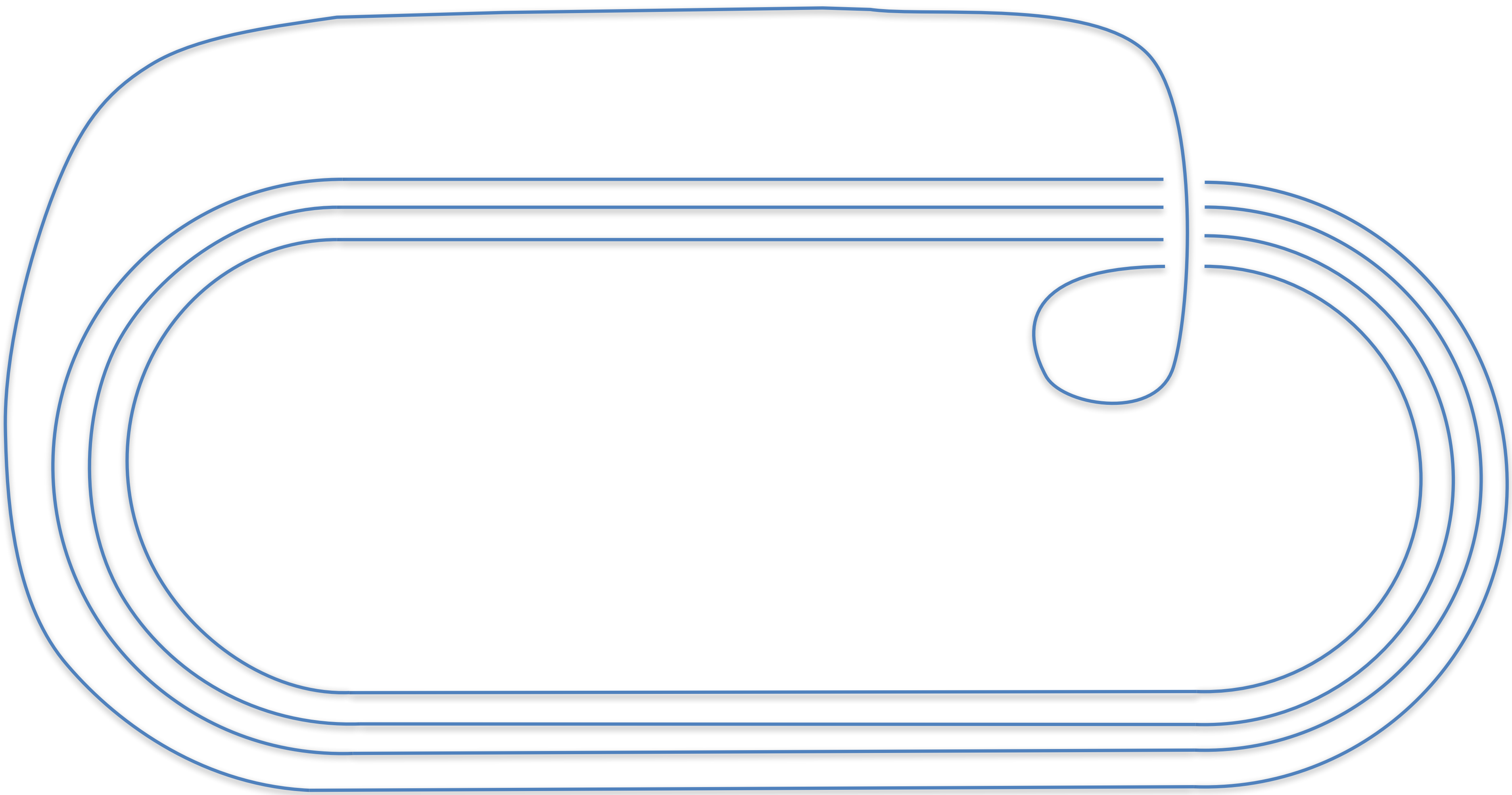


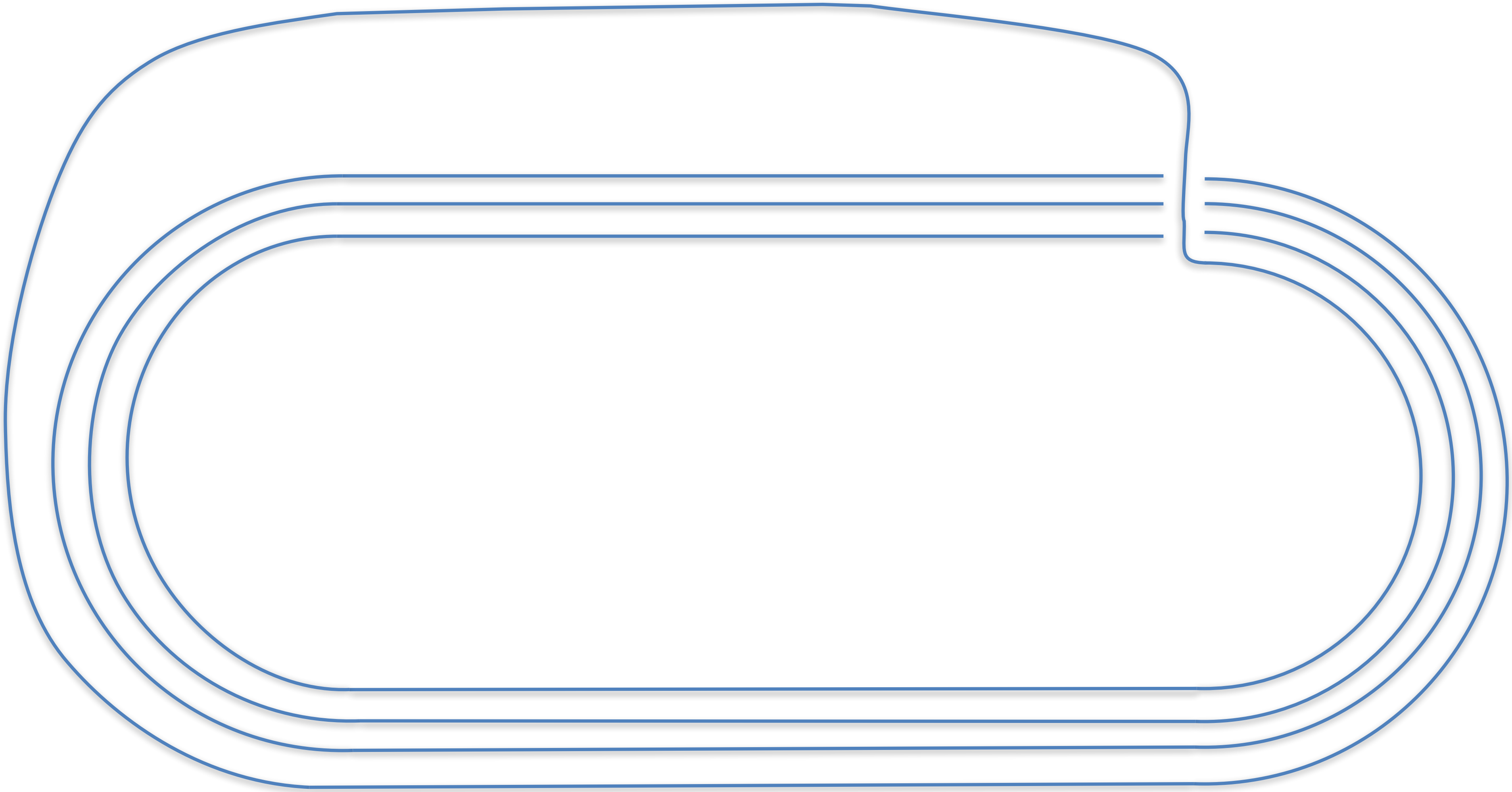


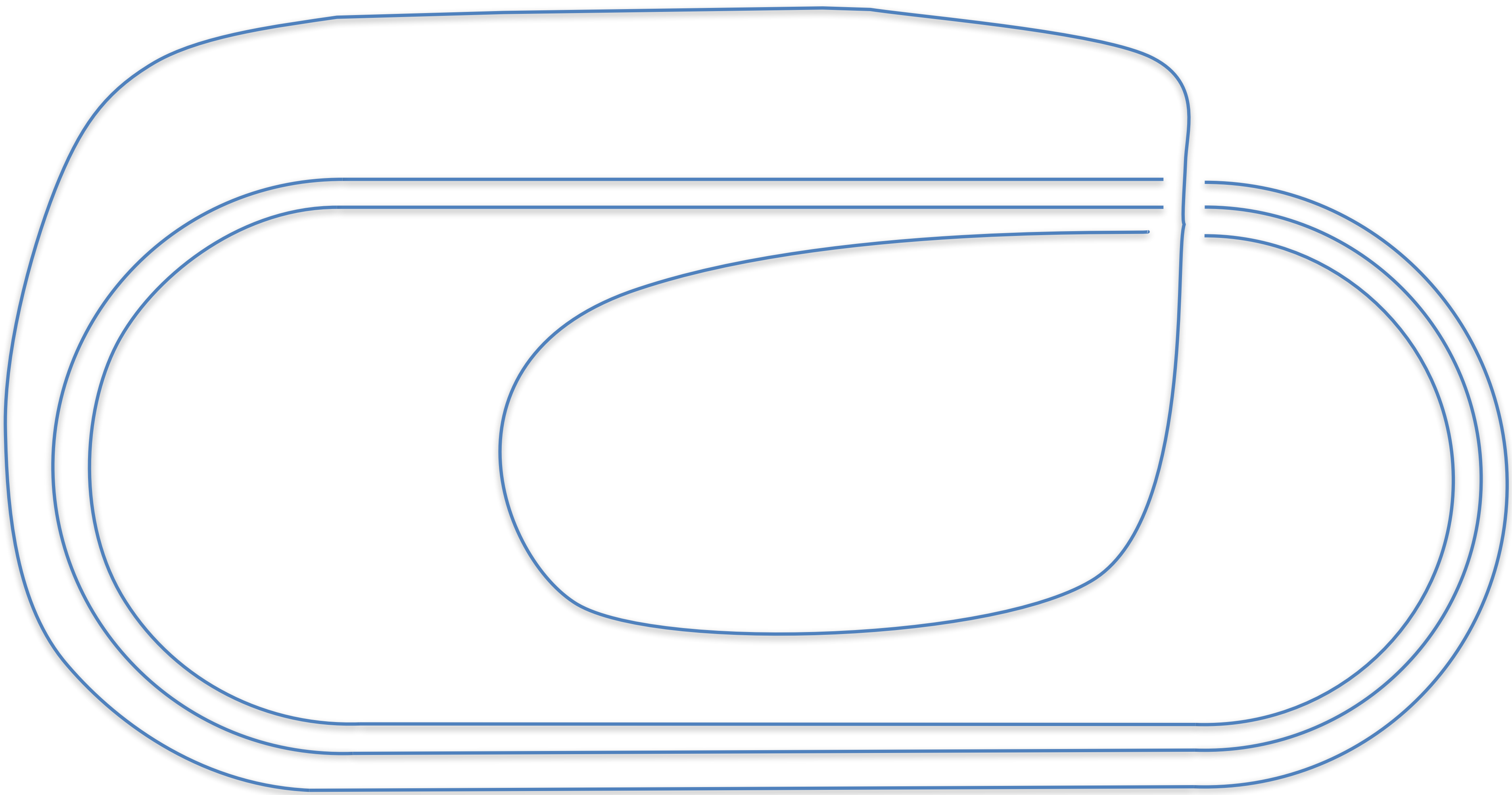


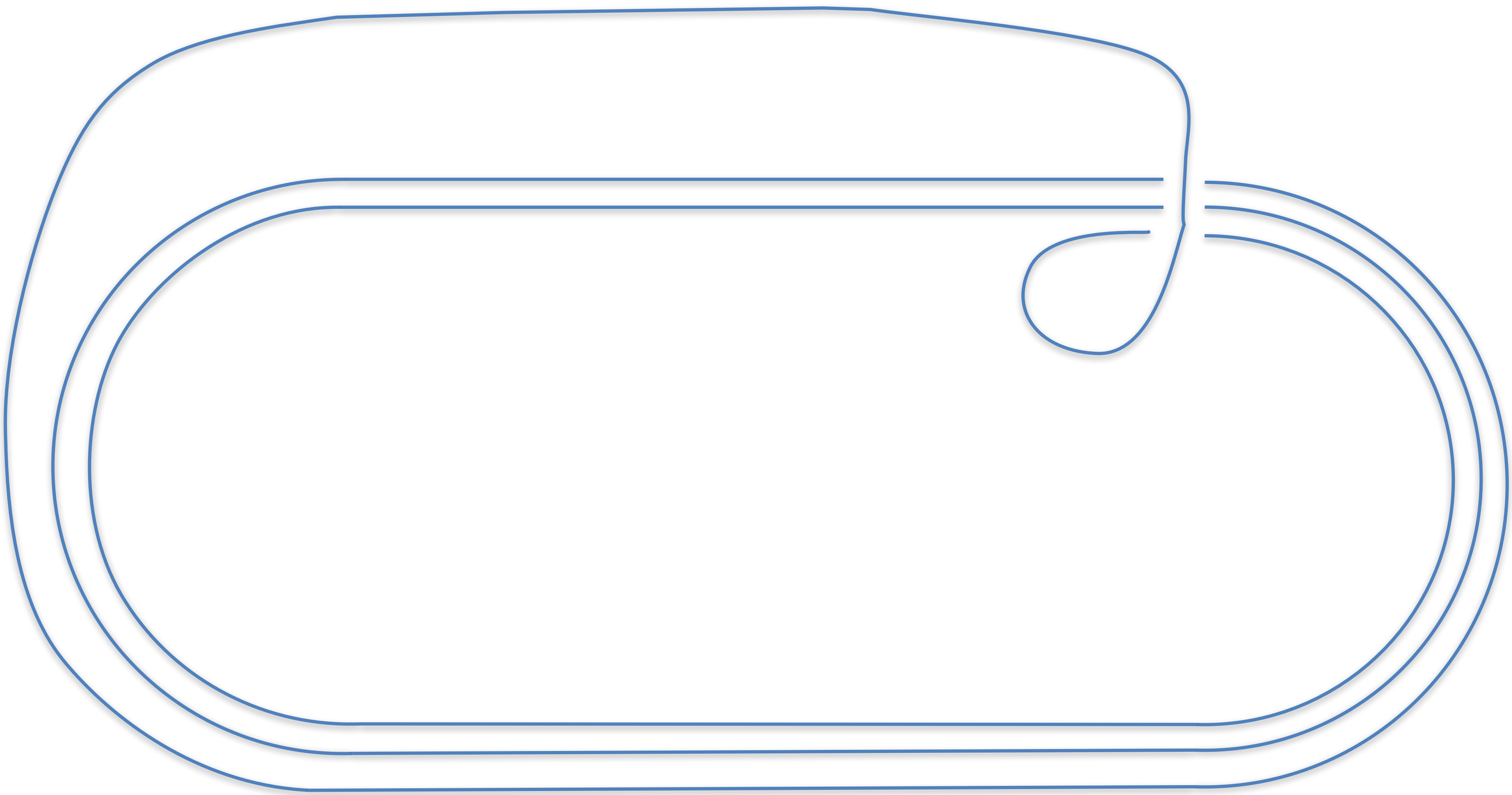


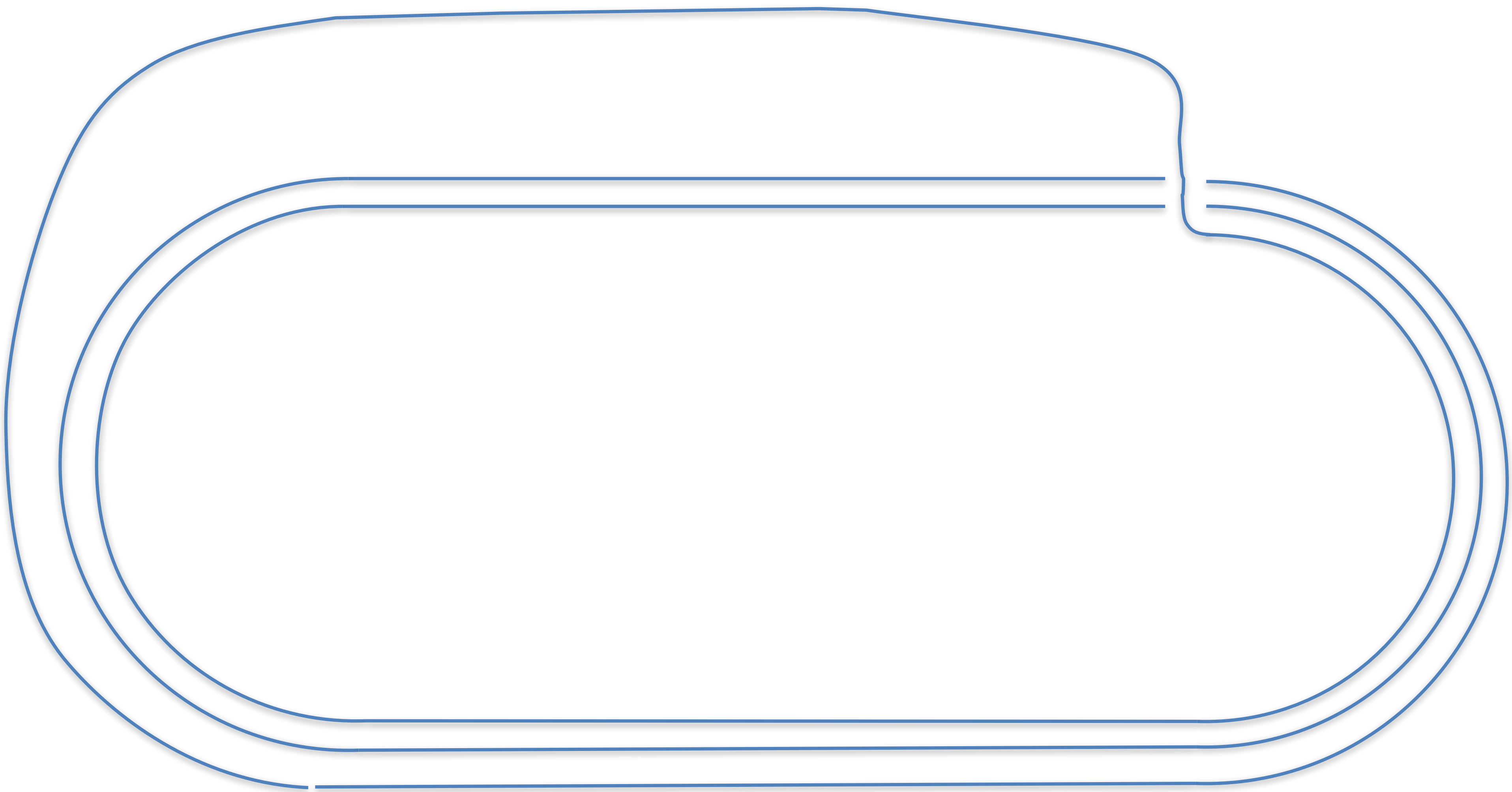




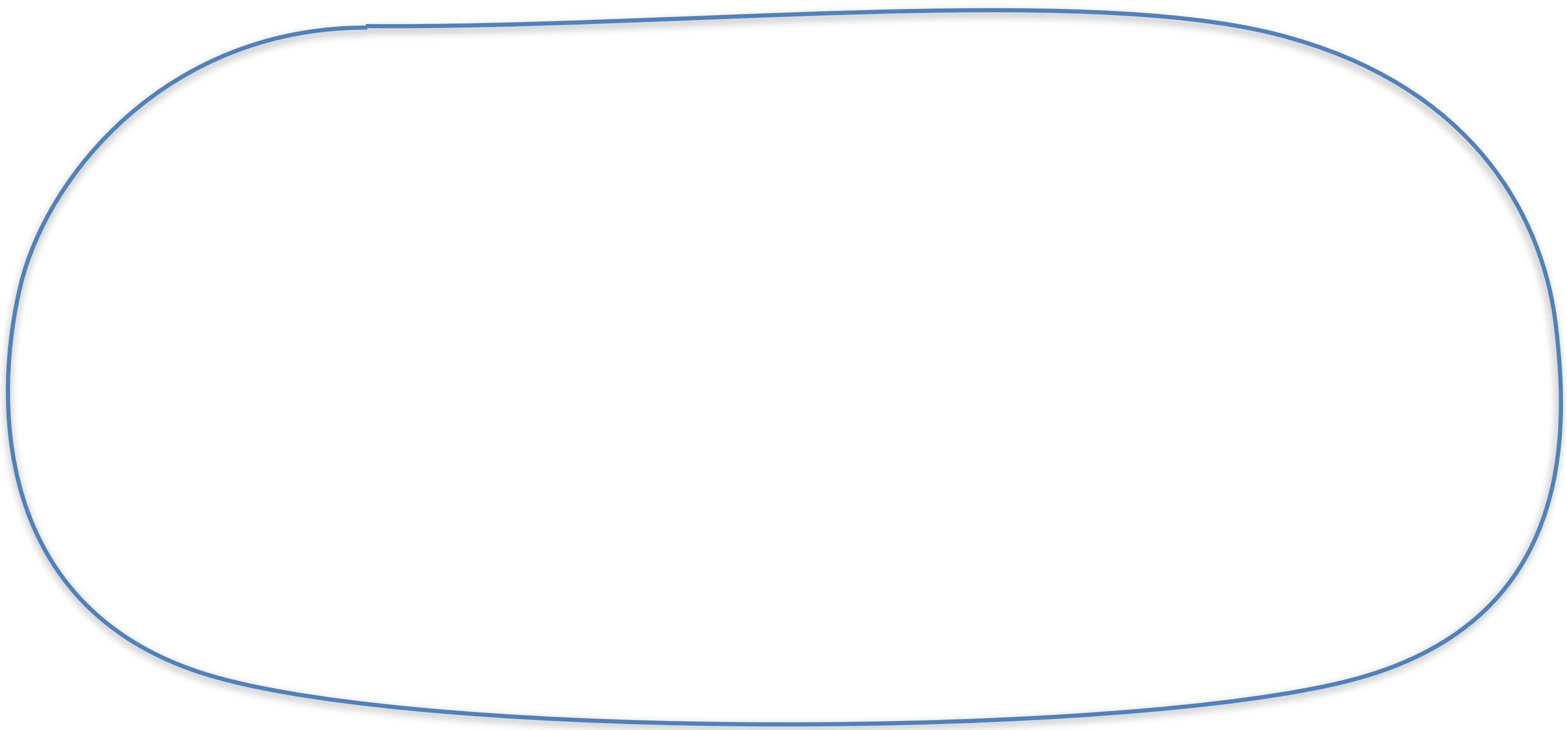




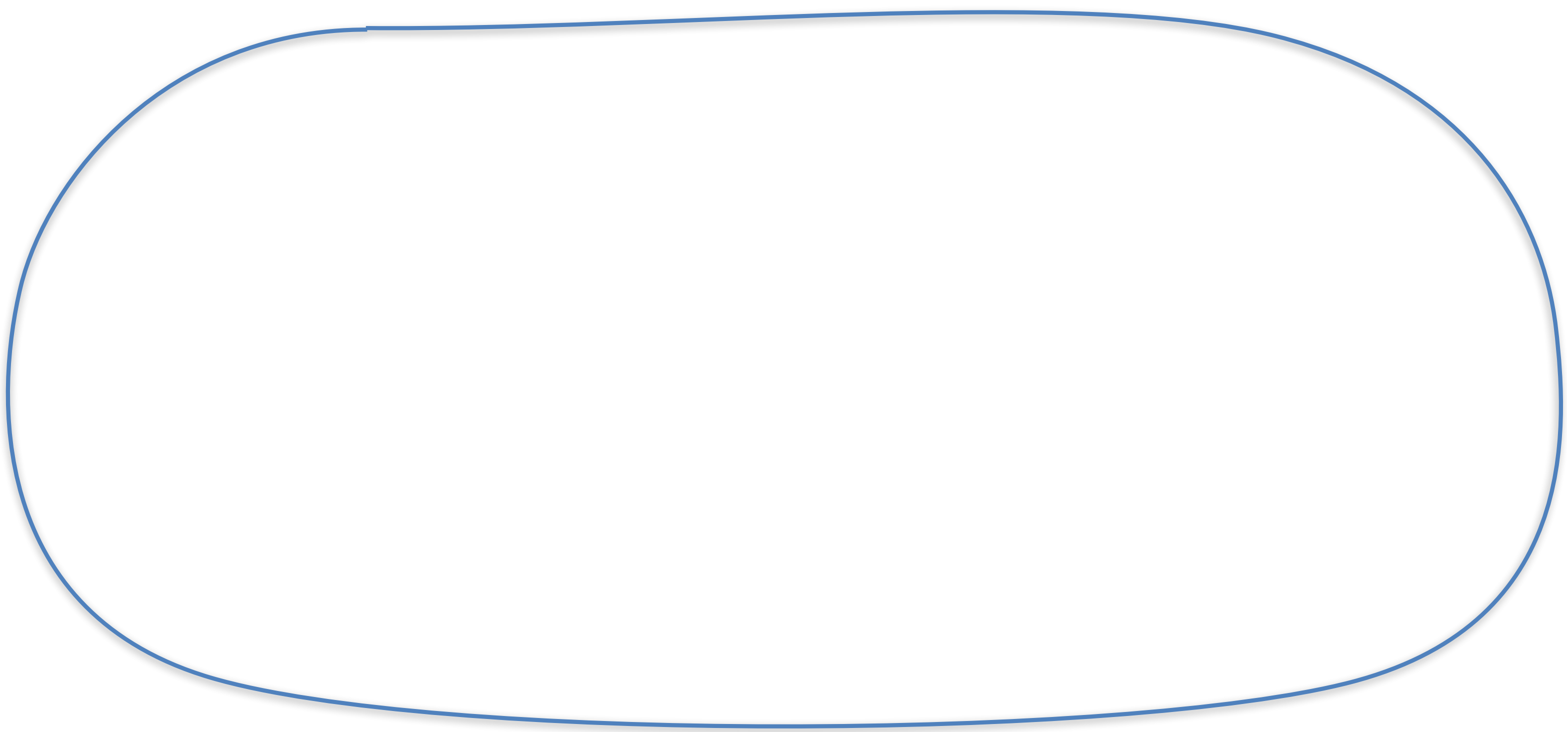




D_0



This particular sequence changed D_n to the trivial Diagram with $2n^2 + 3n$ moves. How do we know that no better, linear sequence of moves exists?



We’ll show that we can improve by at most 2 moves.

An Invariant of Knot Diagrams

We define and use an invariant of knot diagrams to prove lower bounds,

I_{lk} : Diagrams → **Z**

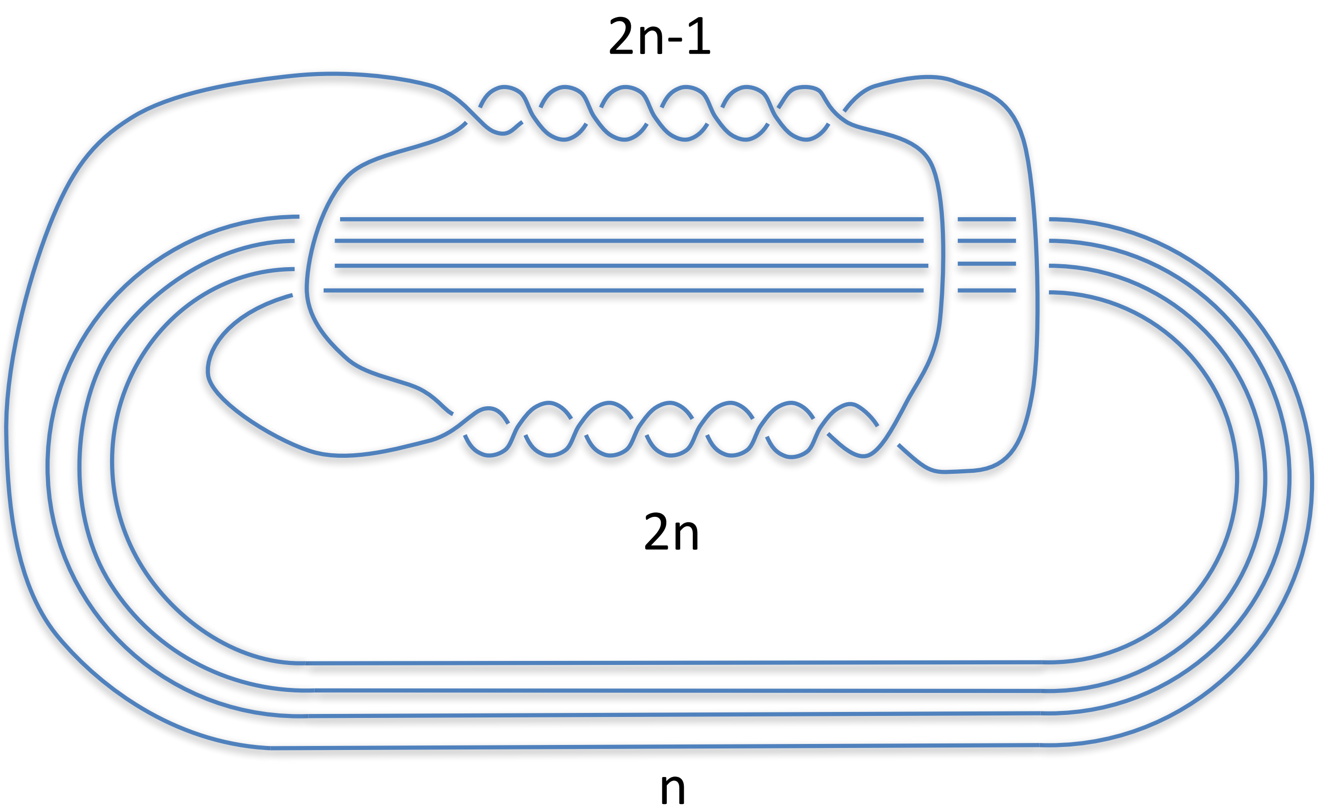
with the following properties:

1. $I_{lk}(\text{O}) = 0$
2. $I_{lk}(\text{D})$ changes by at most 1 under a Reidemeister move.
3. There are n-crossing diagrams D_n of the unknot with $I_{lk}(\text{D}_\text{n}) > n^2/25$ for all n.

Quadratic Lower Bounds

Theorem. The unknot diagram D_n shown below has 7n − 1 crossings and requires no less than *f(n)* Reidemeister moves to transform to the trivial diagram, where *f(n)* satisfies

$$2n^2 + 3n - 2 \leq f(n) \leq 2n^2 + 3n$$



What is the invariant I_{lk} ?

I_{lk} is a finite type invariant for knot diagrams.

Finite Type Invariants for knots were introduced by Vassiliev and Gusarov (1990).

Let V be any invariant of oriented knots with values in **R**.
Extend V to an invariant of singular knots with a single double point by

Extend V to singular knots with m double points by repeating.

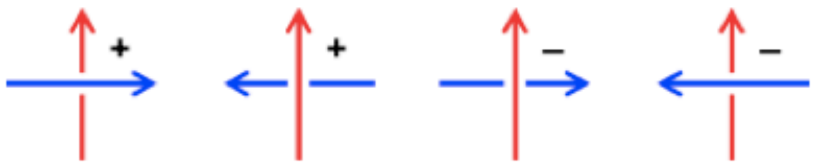
Definition. V has finite type m if its extension to m+1 singular knots is identically zero.

This idea can be used to study general configuration spaces - not just knots.

We use it on the space of diagrams.

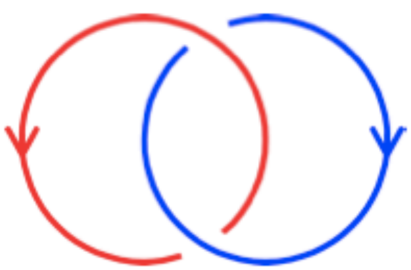
What is the invariant l_{lk}?

Our diagram invariant is based on the linking number. This is obtained from adding the signs of all crossings between two components of a link.

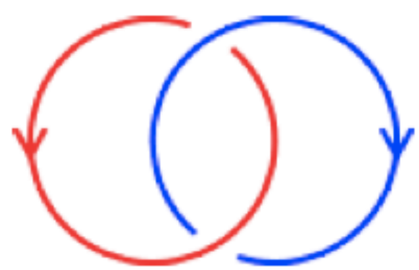


If A and B are the two components, the linking number is defined to be:

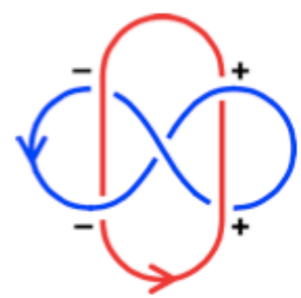
$$lk(A,B) = \frac{1}{2} \sum_{c \in \text{crossings}} \text{sign}(c)$$



-1



$+1$



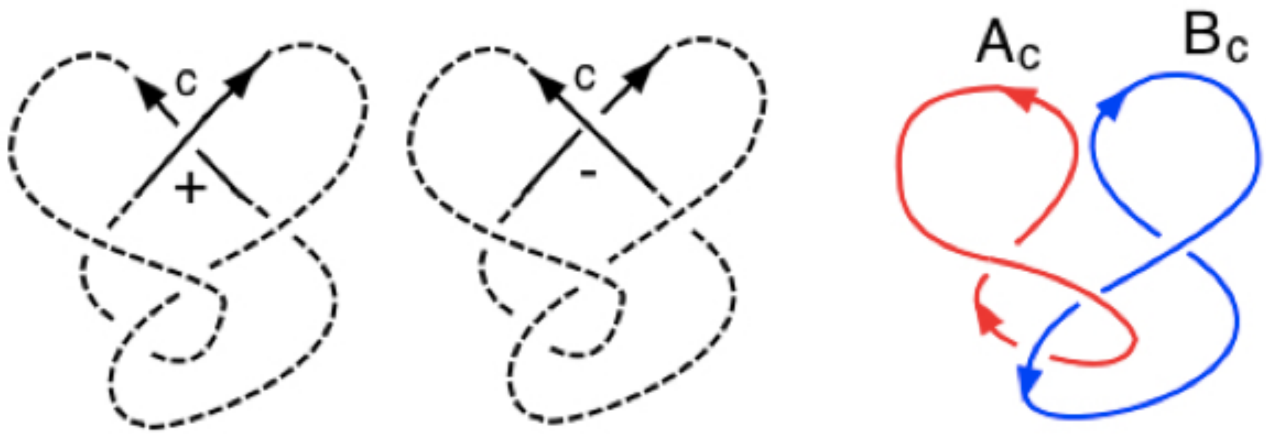
0

What is the invariant I_{lk} ?

Defining I_{lk}
We use a link invariant to define a diagram invariant.

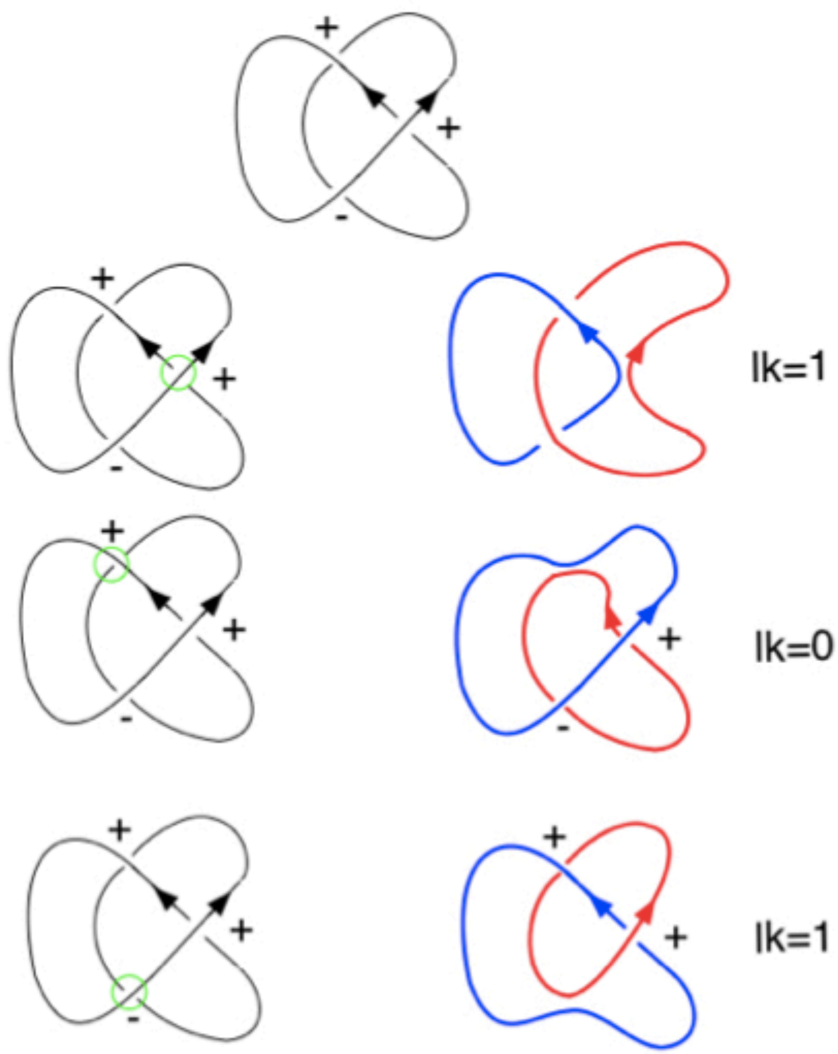
Definition: Let D be a knot diagram. For each crossing c we can smooth the crossing to get a two component link with components A_c and B_c .

$$I_{lk}(D) = \sum_{c \in \text{crossings}} \text{sign}(c)(1 + |lk(A_c, B_c)|)$$



This does *not* give a knot invariant.
It gives a number that changes under a Reidemeister move.

Computing I_{lk}

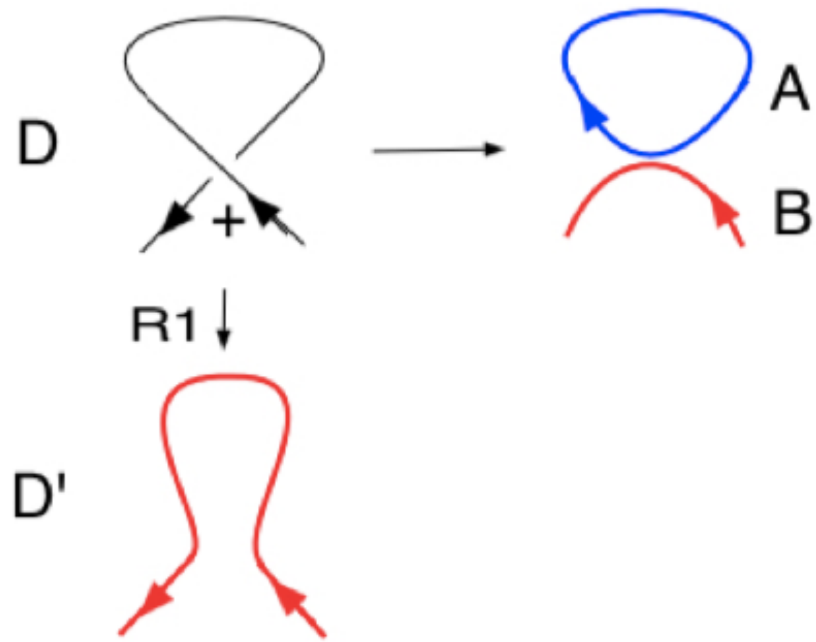


$$I_{lk} = \sum_{\text{crossings}} \text{sign}(c)(1+|lk(A_c, B_c)|) = 1(1+1)+1(1+0)-1(1+1) = 1$$

How I_{lk} changes

How does I_{lk} change under a Reidemeister move taking D to D_0 ?

Lemma: $| I_{lk}(D_0) - I_{lk}(D)| \leq 1$.
Proof: Check the change in Q for each Reidemeister move.

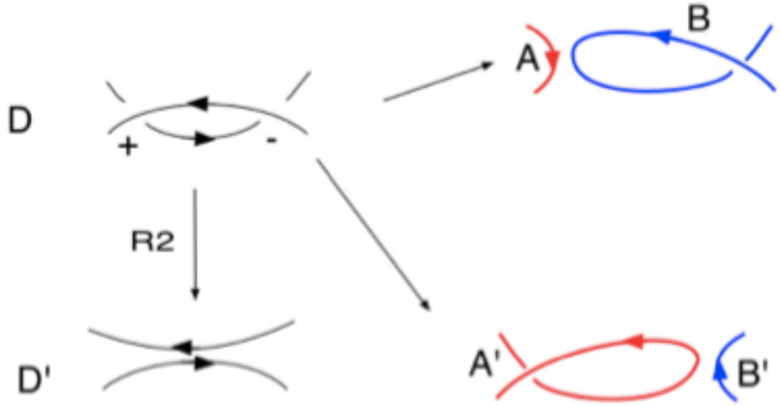


$I_{lk}(D) = I_{lk}(D') + (|lk(A, B)| + 1) = I_{lk}(D') + 1$

$I_{lk}(D)$ changes by 1.

How I_{lk} changes

How does I_{lk} change under a Reidemeister II move taking D to D' ?

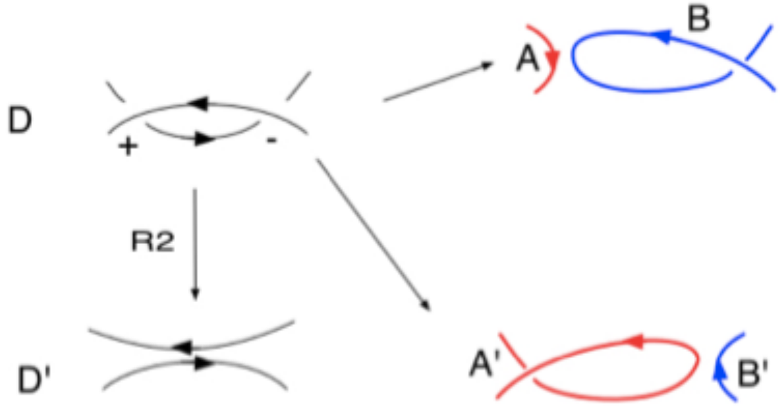


I_{lk}(D) doesn't change.

$$I_{lk}(D) = I_{lk}(D') + (|lk(A, B)| + 1) - (|lk(A', B')| + 1) = I_{lk}(D').$$

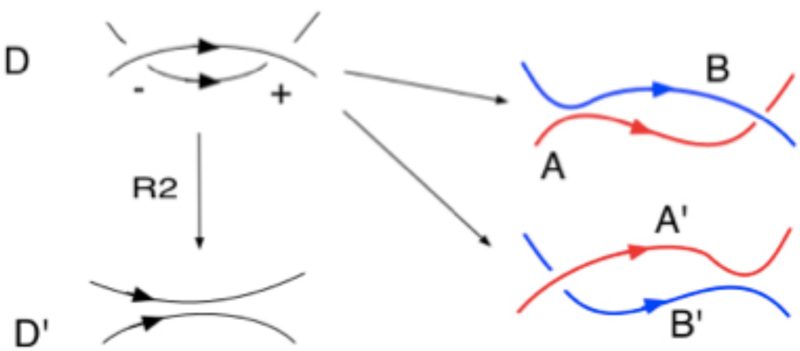
How I_{lk} changes

How does I_{lk} change under a Reidemeister II move taking D to D’ ?



I_{lk}(D) doesn’t change.

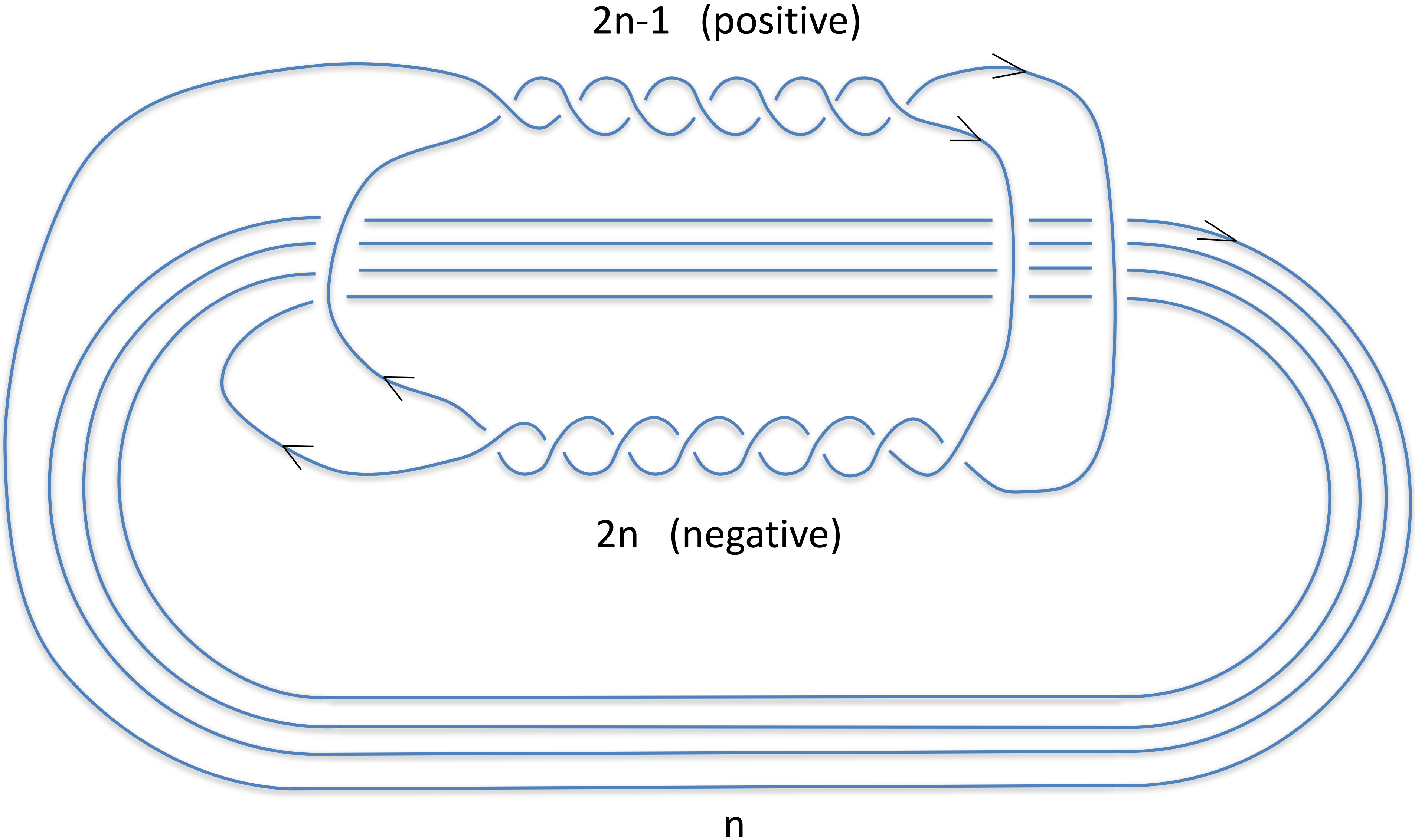
$I_{lk}(D) = I_{lk}(D’) + (|lk(A, B)| + 1) - (|lk(A’, B’)| + 1) = I_{lk}(D’).$



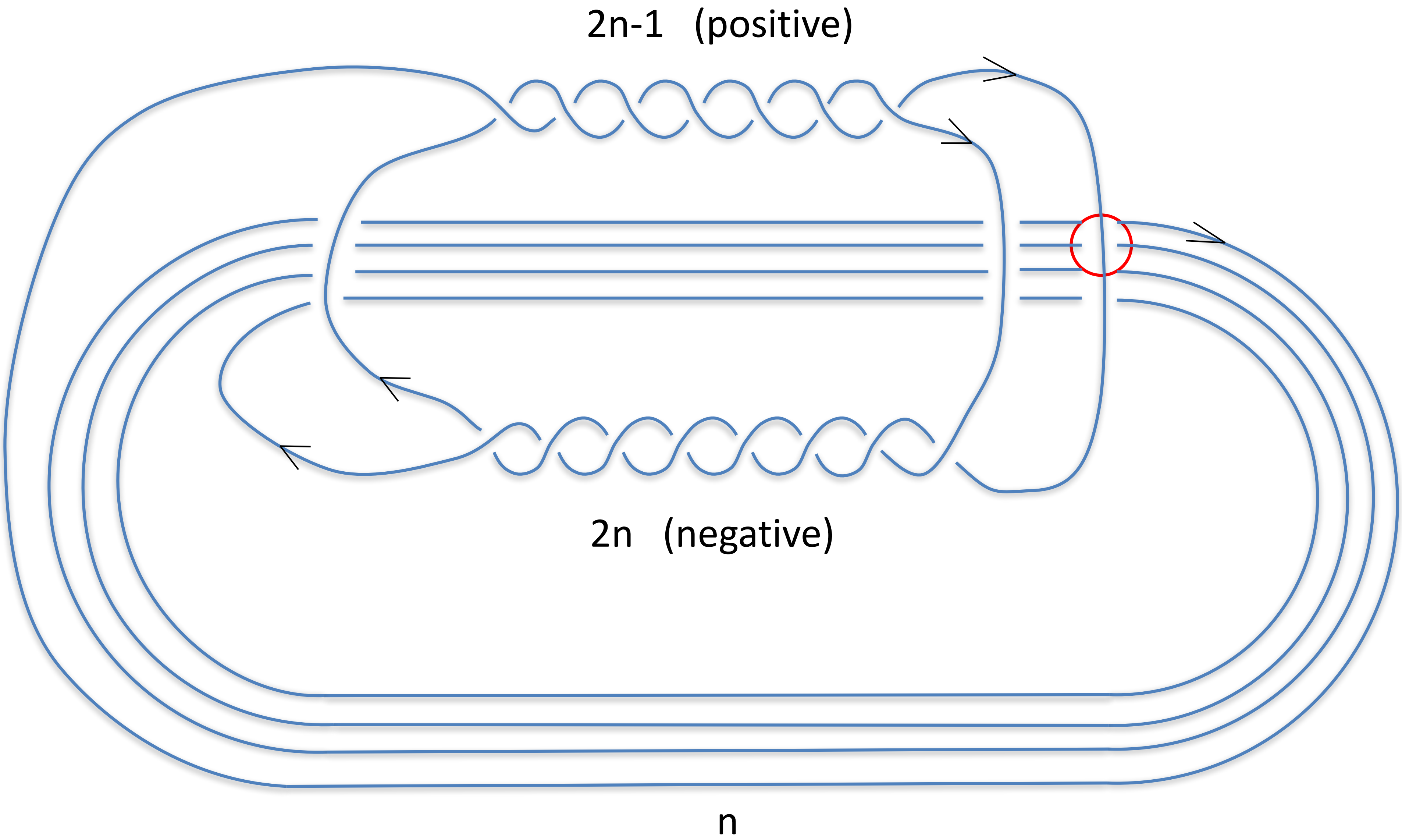
I_{lk}(D) changes by ± 1.

$I_{lk}(D) = I_{lk}(D’) - (|lk(A, B)| + 1) + (|lk(A’, B’)| + 1) = I_{lk}(D’) \pm 1$

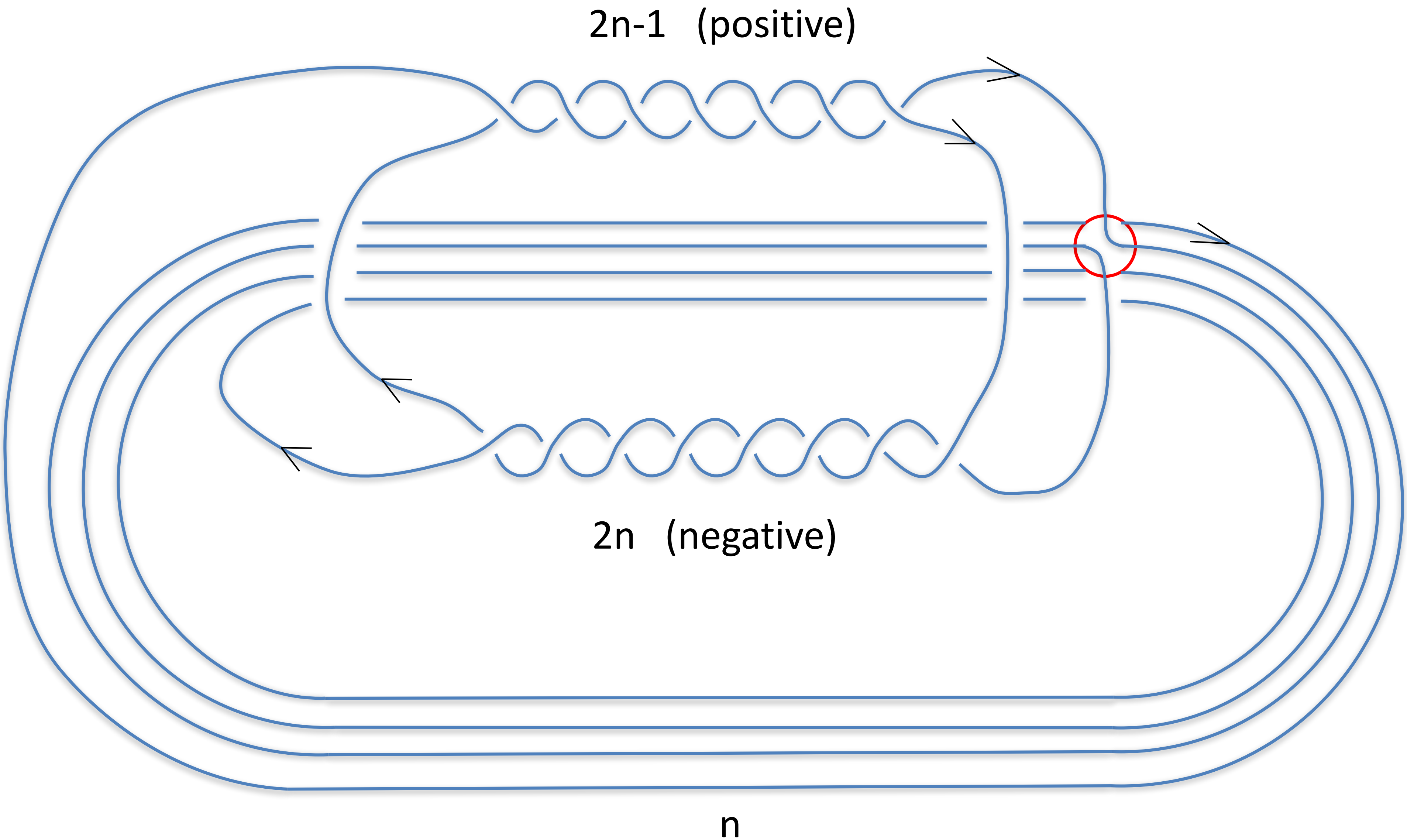
Example of computation l_{lk} at one crossing:



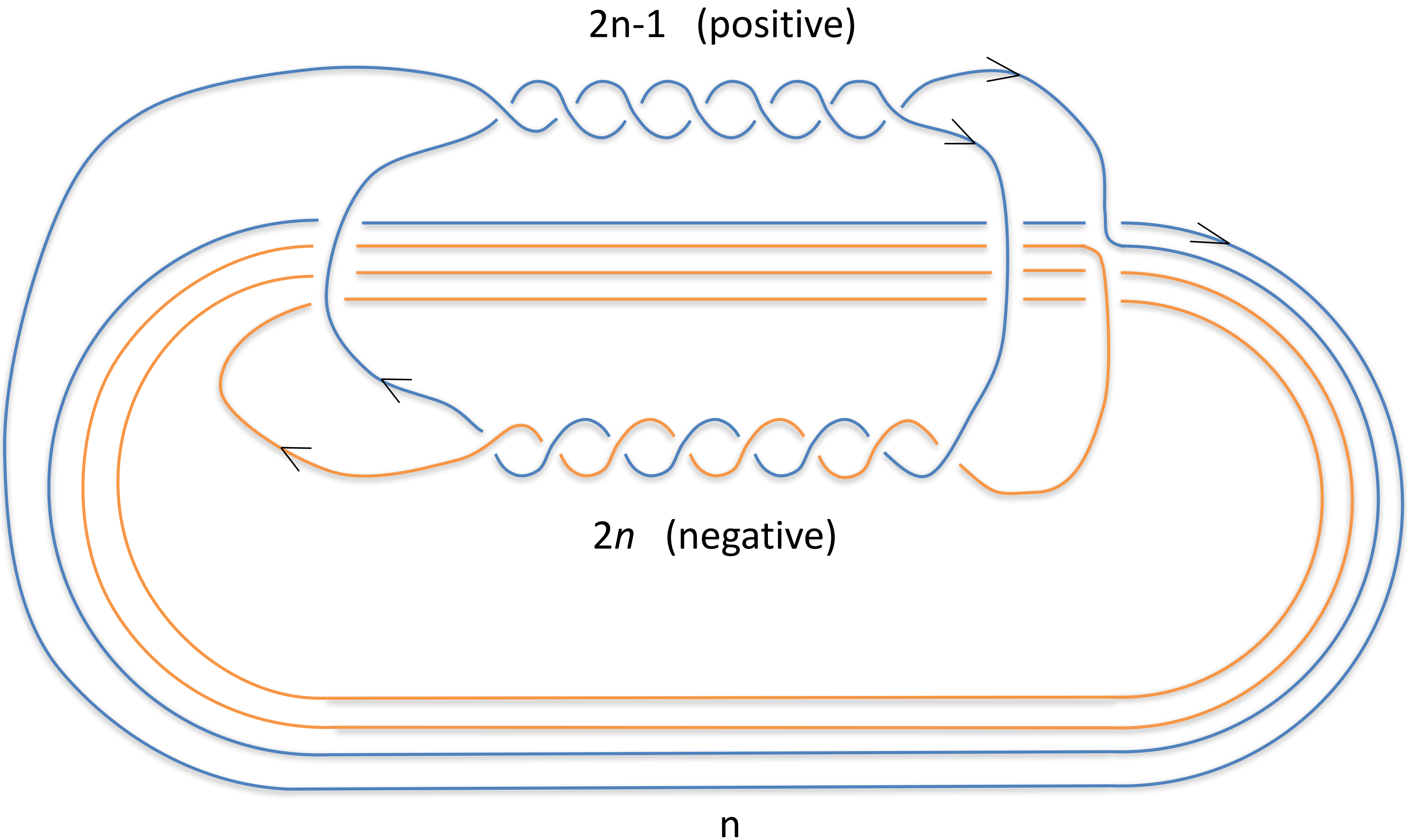
Example of computation l_{lk} at one crossing:



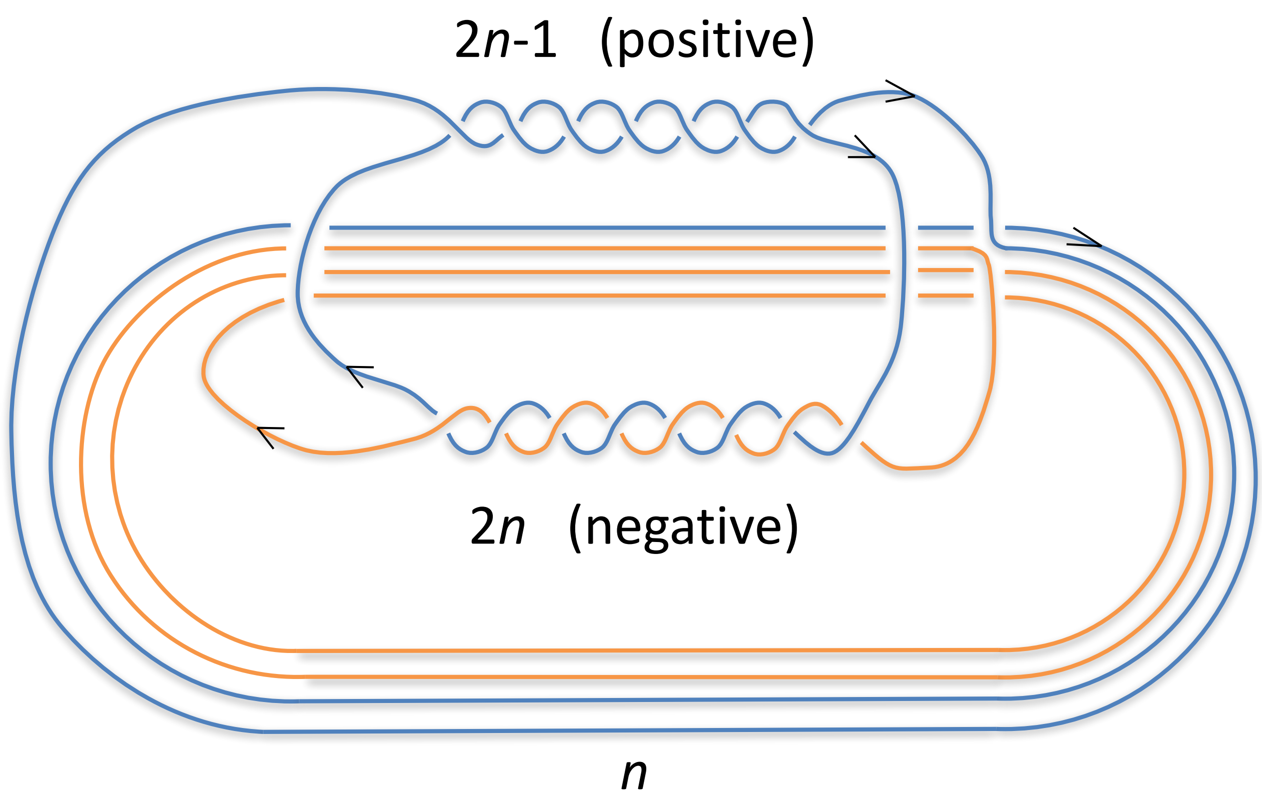
Example of computation I_{lk} at one crossing:



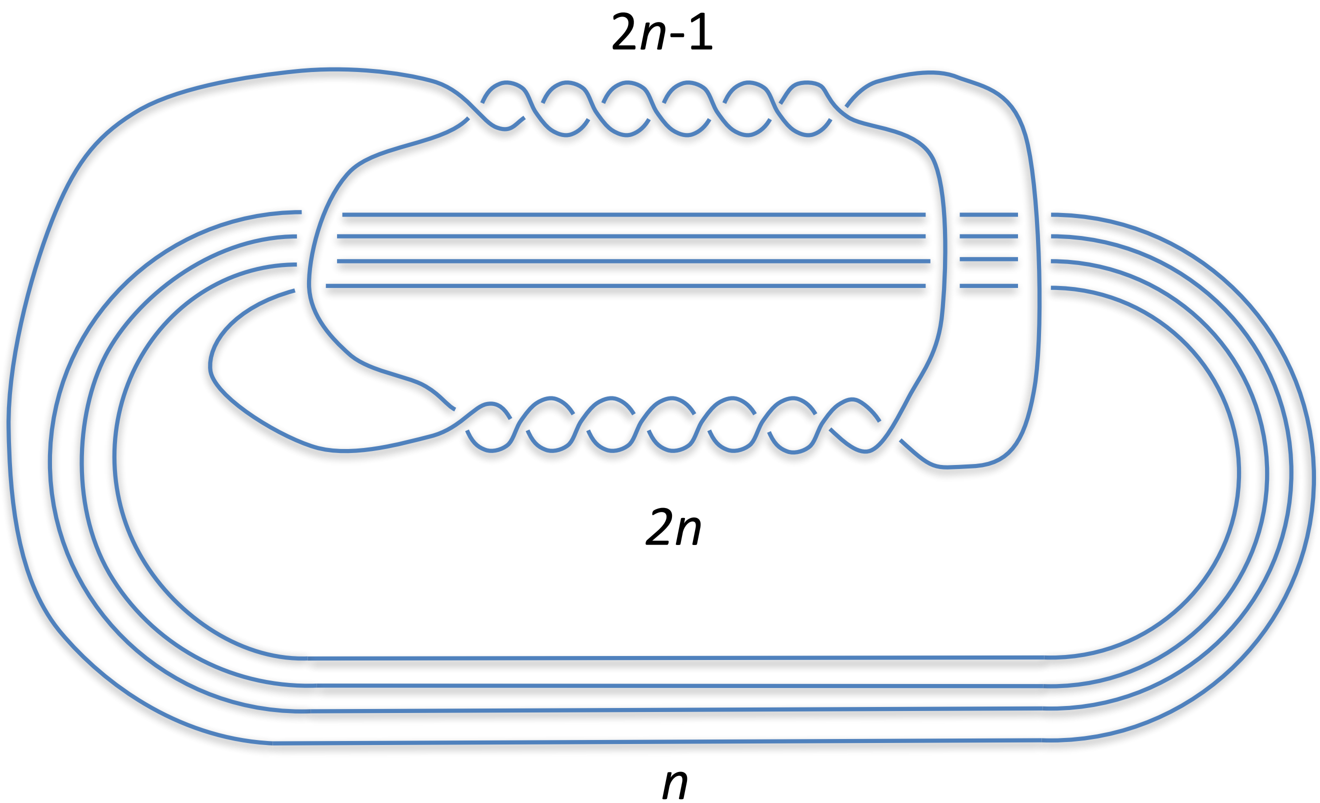
Example of computation I_{lk} at one crossing:



Example of computation I_{lk} at one crossing:



The crossing was positive and the linking number is $-n$ so this crossing contributes $(+1)(1+|-n|) = 1+n$.

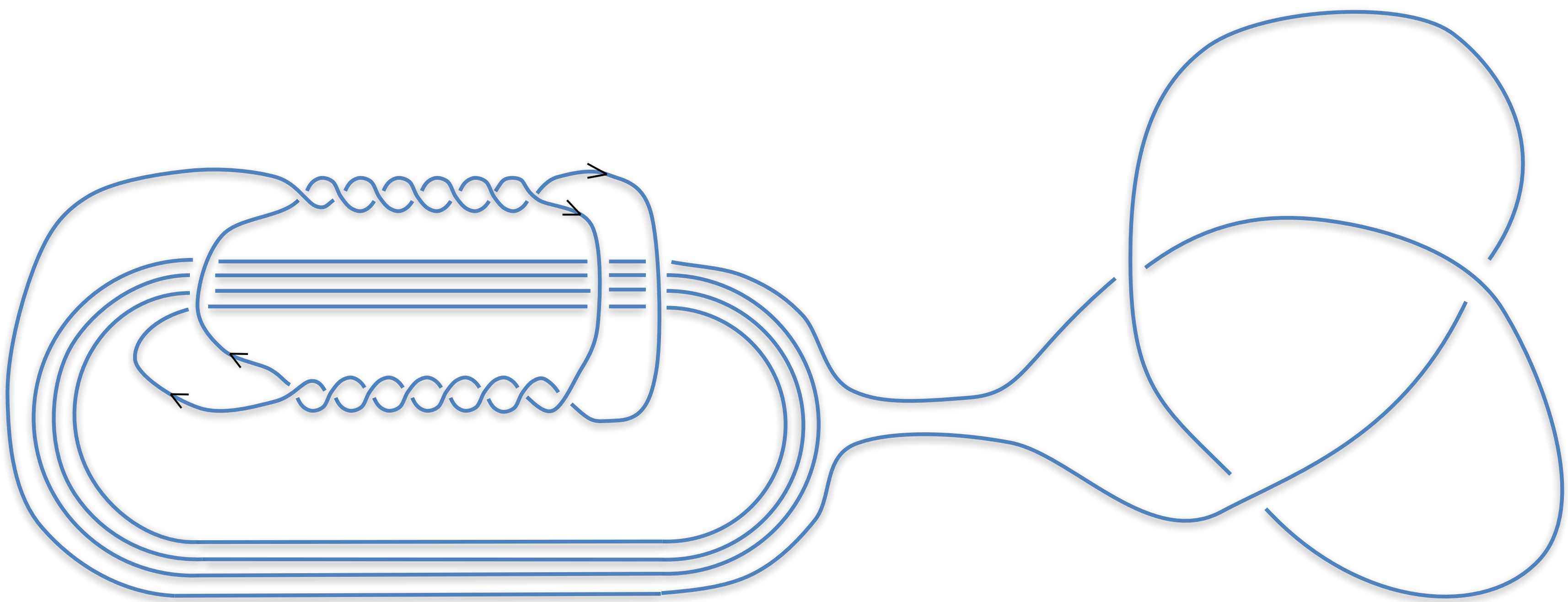


Sum all contributions from all crossings:

$I_{lk}(D_n) = 2n^2 + 3n - 2$

Since *any* Reidemeister move changes the value of $I_{lk}(D_n)$ by at most 1, and D_n has $7n$-1 crossings, we get a quadratic lower bound on the number of required moves.

A similar lower bound is obtained for any knot type by connected sum with a fixed diagram:



A better than quadratic lower bound cannot be established using the invariant I_{lk}.

Open Problem: Is there a knot diagram for the unknot that requires $O(n^3)$ Reidemeister moves to transform to a trivial diagram?

Computation and complexity have many still undiscovered connections to Topology

There are many intriguing possibilities for applying ideas from approximation algorithms, probabilistic algorithms, quantum computing etc to questions in topology and geometry.

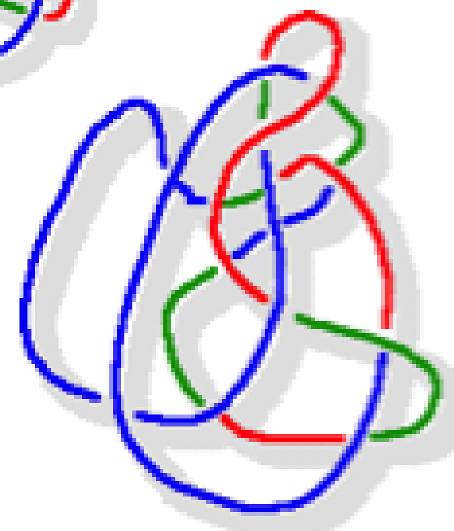
Zero Knowledge Proofs

Example

There is a zero knowledge proof that this knot is non-trivial.

The technique applies to a wide variety of knots.
The ideas are due to
Goldwasser, Micali, Rackoff (1985)
Goldreich, Micali, Wigderson (1991)

3 colorings

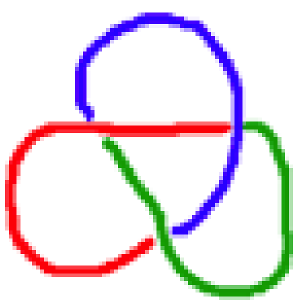


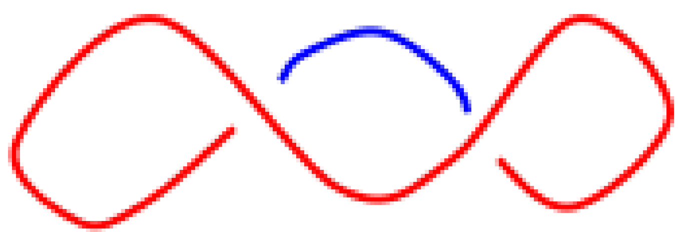
A knot is **3-colorable** if we can

1. Color each strand
2. Use all 3 colors
4. 1 color or 3 colors at each crossing.

Strongly 3-colorable:
All 3 colors appear at each crossing.

Theorem: Tricolorability implies the knot is non-trivial

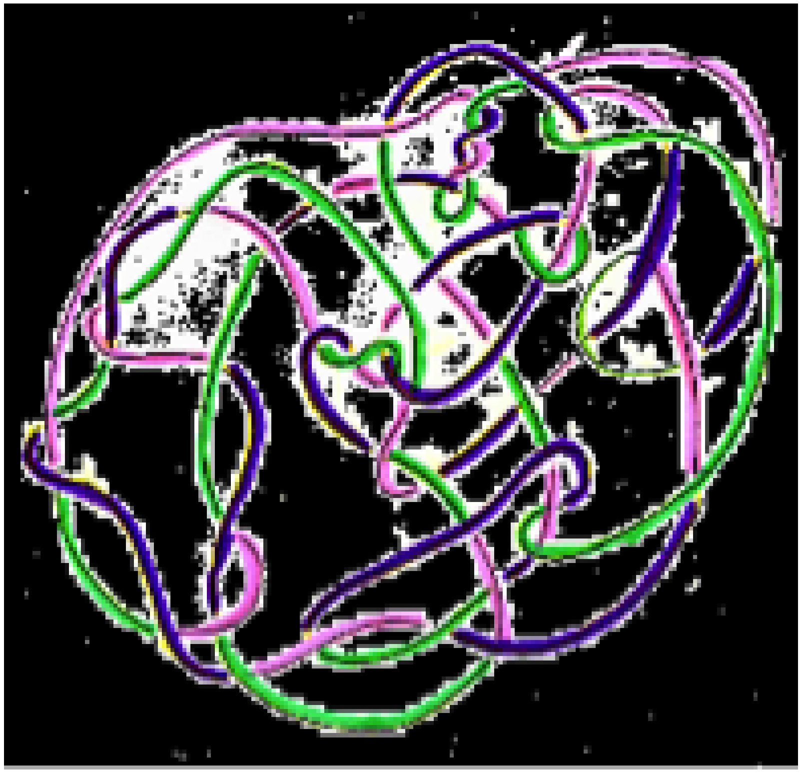




Strong 3-colorings

A knot has a strong 3 coloring if all three colors appear at each crossing.

A knot with a strong 3-coloring is non-trivial.



This diagram has a strong 3-coloring.

RGB

RBG

GBR

GRB

BRG

BGR

There are six different 3-colorings of *K*, using six permutations of RGB.
I create six copies, using all six.

Zero Knowledge Proofs

Example

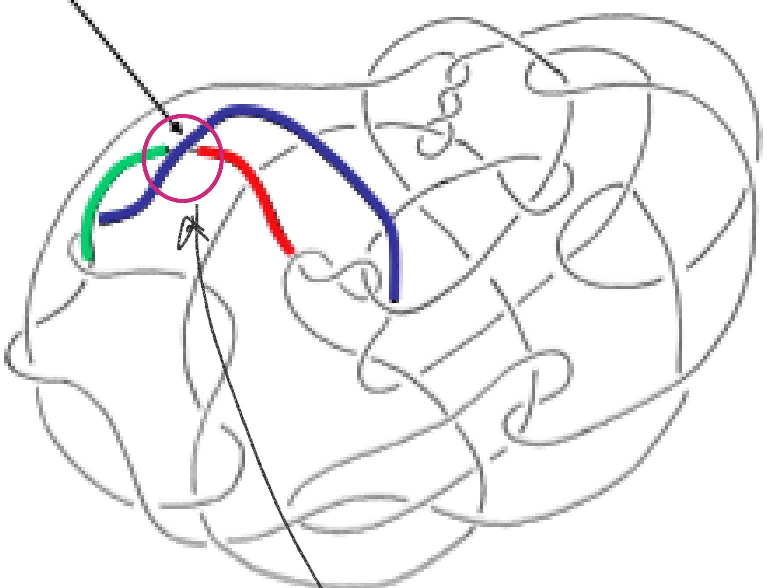
There is a zero knowledge proof that this knot is non-trivial.

We will play a game that will convince you I know how to strongly 3-color this knot.

At each step,

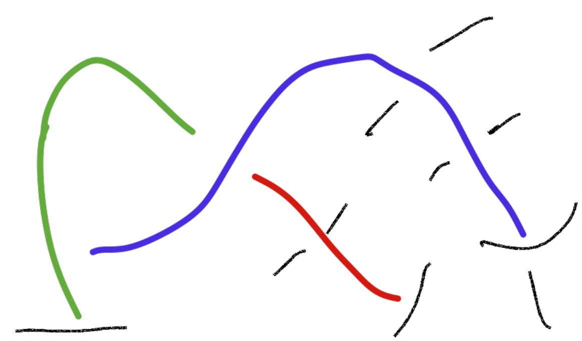
1. I randomly select one of my six colorings and put it on the table in front of you with masking tape hiding the colors. You see the above, with no colors.
2. You select a crossing.
3. I peel the tape from the three strands of the knot meeting that crossing.
4. We repeat this procedure.

Zero Knowledge Proof



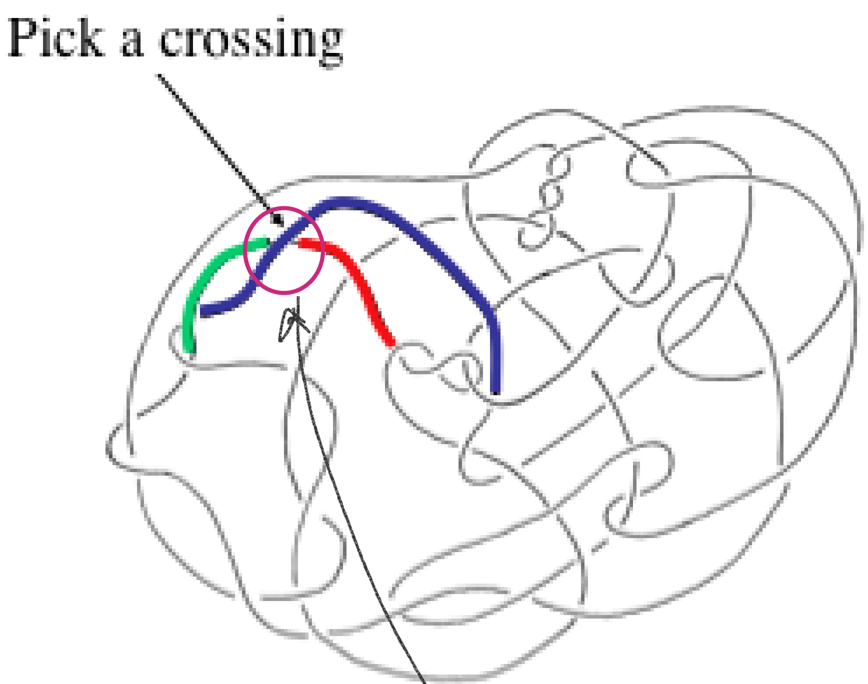
Pick a crossing

CLAIM
This is a non-trivial knot

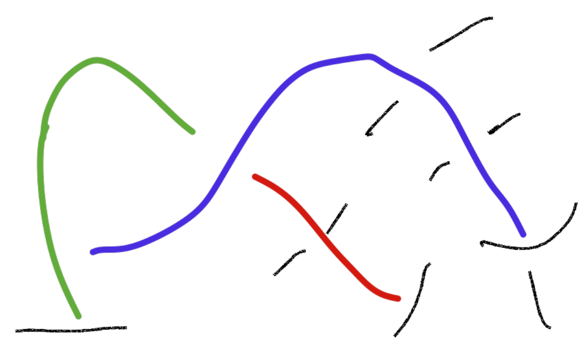


What you see

Zero Knowledge Proof

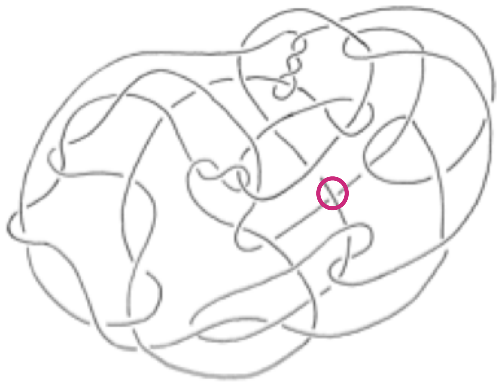


CLAIM
This is a non-trivial knot



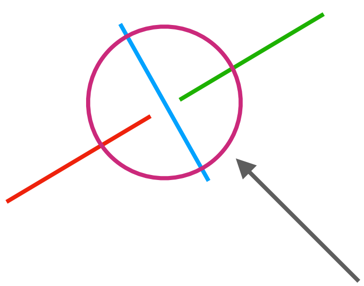
What did you learn?

Nothing!



Sequence of moves:

1. I roll a die and randomly select one of the six color permutations. (You don't know which one.) I cover it up and put it in front of you uncolored.
2. You select a crossing.
3. I uncover the crossing and the color of adjacent edges.



You see this. There are three colors near the vertex.

Hidden 1

Hidden 2

Hidden 3

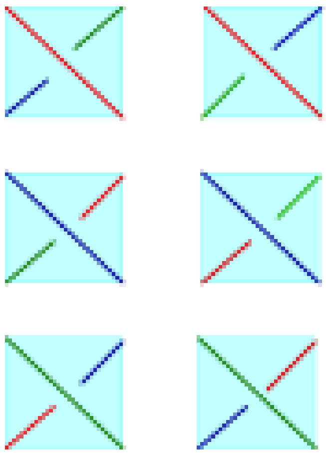
Hidden 4

Hidden 5

Hidden 6

The six 3-colorings of K using 6 permutations of RGB

Zero Knowledge Proof

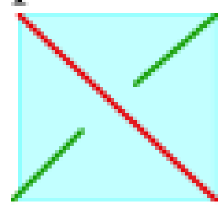


Procedure: I make six copies of the knot, colored with each of the six permutations of 3 colors.

Repeat the following process:

1. I shuffle the six copies of the knot
2. You pick one of the six copies
3. You pick a crossing
4. I uncover the strands at that crossing
5. Repeat

If I know a strong 3-coloring you always see one of the above pictures. If I am cheating, you will with probability $p>0$ catch me and see something like this:



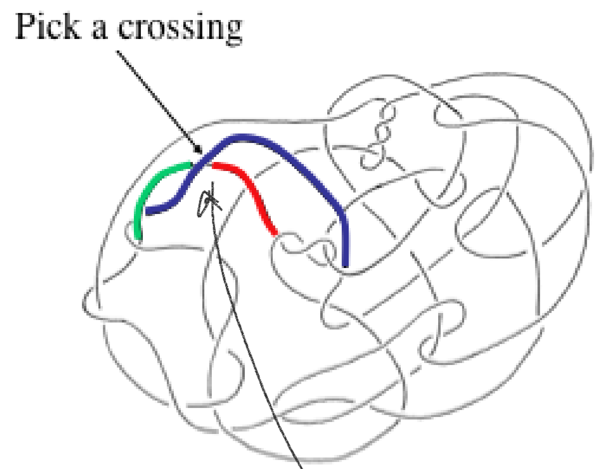
With many repetitions, the probability of successful cheating approaches zero.

Zero Knowledge Proofs

There is a zero knowledge proof that this knot is non-trivial.

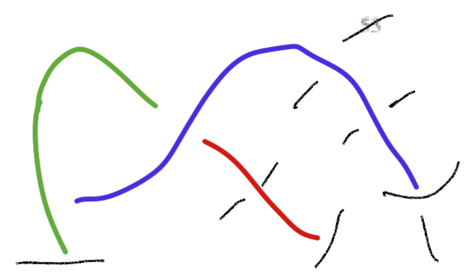
1. I will convince you that this knot is non-trivial.
2. You learn nothing about the proof. You will not be able to prove to someone else that the knot is non-trivial, even though you knot that it is with probability *p* that approaches 1 exponentially fast as we repeat the game.

Zero Knowledge Proof



CLAIM
This is a non-trivial knot

What did you learn?



Nothing!

I hope that everyone learned something this week.

Thank you for listening.