From Isometric Embeddings to C^1 Fractals

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Keith Arnold

Outline

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Some Good Reasons Why Spheres Are Rigid

As convex surfaces.

Theorem (Cohn-Vossen (C^ω) 1936 ; Herglotz (C^3) 1943 ; Sacksteder (C^2) 1962 ; Pogorelov (-), 1973)

Any two isometric compact closed convex surfaces in $\mathbb{E}^3$ are congruent.

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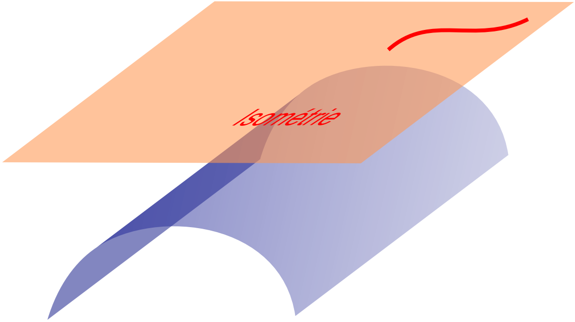

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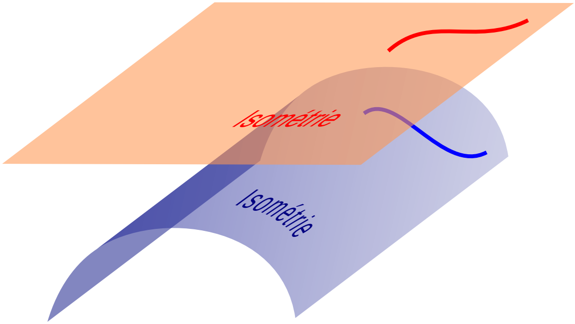

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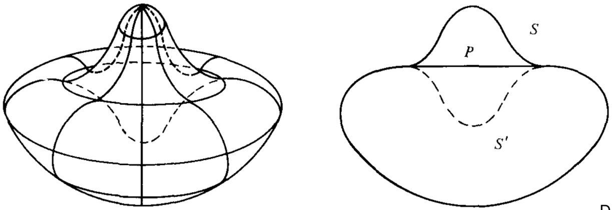

Do Carmo

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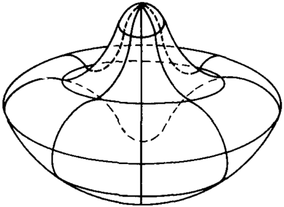

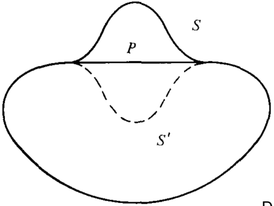

Do Carmo

Corollary

The sphere has no flex: any infinitesimal isometric deformation has positive curvature, hence is a round sphere.

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As a “non-reducible” surface.

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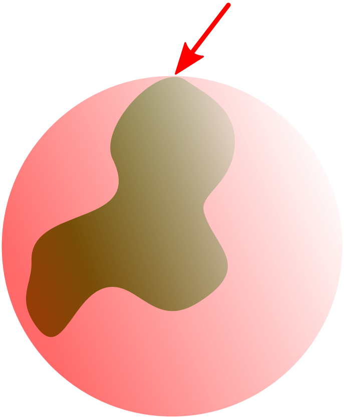

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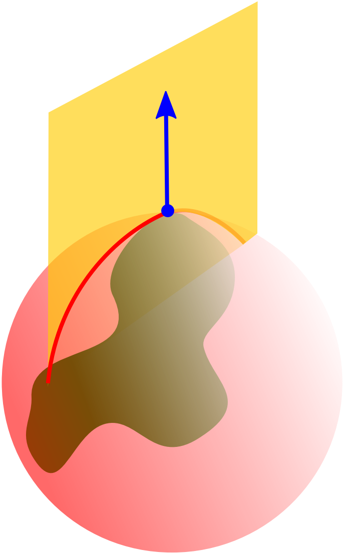

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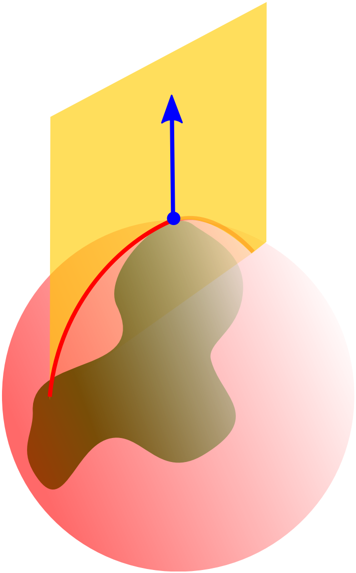

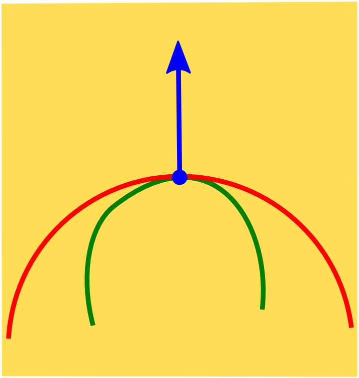

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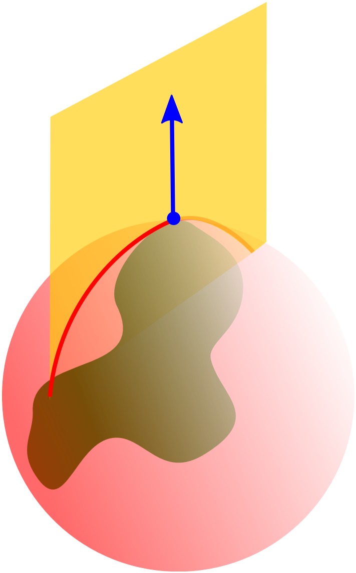

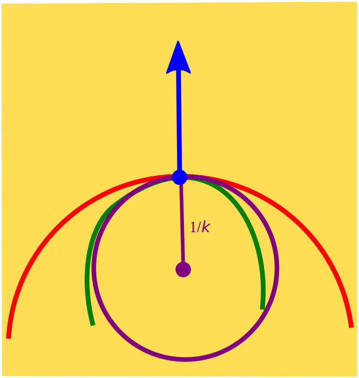

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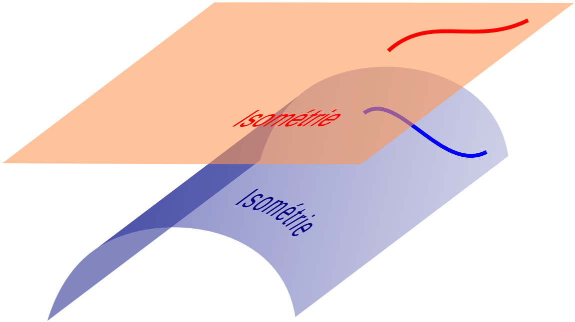
Isometric surfaces in $\mathbb{E}^3$ have the same Gaussian curvature.



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Theorem (Nash 1954, Kuiper 1955)

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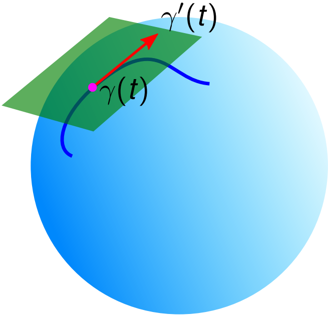
Connelly 1993 (Handbook of convex geometry)

"I know of no explicit construction of such a flex or even of an explicit C^1 embedding other than the original sphere."

From Riemann to Nash

Riemann 1854
(On The Hypotheses Which Lie At The Bases Of Geometry)

"…and consequently *ds* is the square root of an always positive integral homogeneous function of the second order of the quantities *dx*…"



$$\ell(\gamma) = \int_I \sqrt{g(\gamma'(s), \gamma'(s))}ds$$

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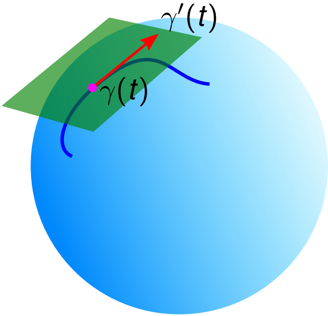

$$\begin{aligned} \ell(\gamma) &= \int_I \sqrt{g(\gamma'(s), \gamma'(s))}\, ds \\ &= \int_I \sqrt{\langle \gamma'(s), \gamma'(s) \rangle_{\mathbb{E}^3}}\, ds \end{aligned}$$

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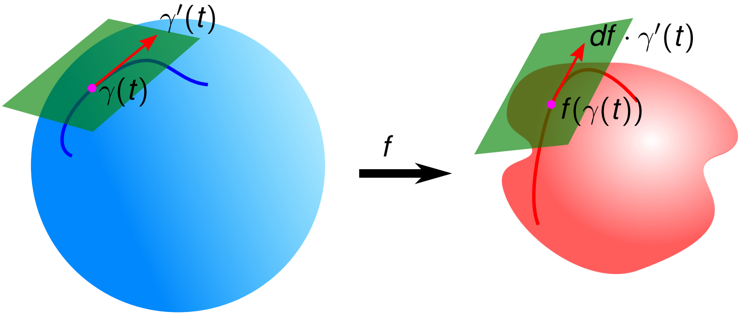

$$\forall \gamma : \ell(\gamma) = \ell(f \circ \gamma) \Leftrightarrow \forall u, v \in T_{\gamma(t)}\mathbb{S}^2 : g(u, v) = \langle df.u, df.v \rangle_{\mathbb{E}^3}$$

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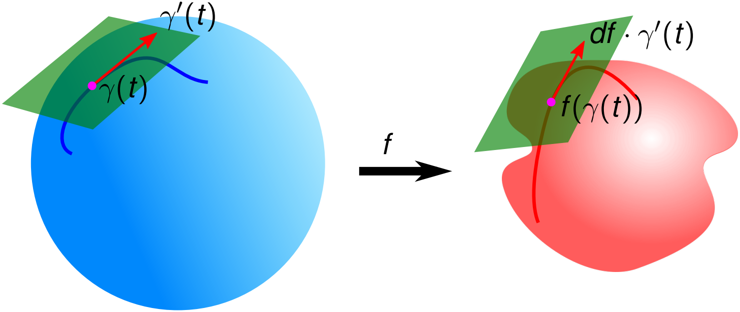

$$\Leftrightarrow \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle_{\mathbb{E}^3} = E, \quad \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle_{\mathbb{E}^3} = F, \quad \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right\rangle_{\mathbb{E}^3} = G$$

The isometric embedding problem

Find $f : (M^n, g) \to \mathbb{E}^s$ s.t. $\forall p \in M, \forall u, v \in T_pM$:

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C^1 ISOMETRIC IMBEDDINGS

By John Nash

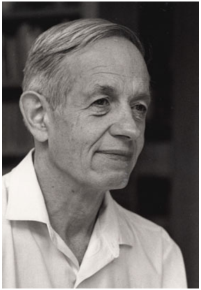
(Received February 26, 1954)
(Revised June 21, 1954)

Introduction

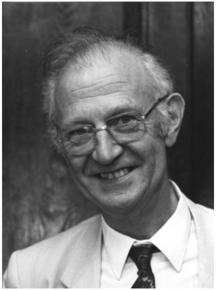
The question of whether or not in general a Riemannian manifold can be isometrically imbedded in Euclidean space has been open for some time. The local problem was discussed by Schlaefli [1] in 1873 and treated by Janet [2] and Cartan [3] in 1926 and 1927.

This question comes up in connection with the alternative extrinsic and intrinsic approaches to differential geometry. The historically older extrinsic attitude sees a manifold as imbedded in Euclidean space and its metric as derived from the metric of the surrounding space. The metric is considered to be given abstractly from the intrinsic viewpoint.

This intrinsic approach has seemed the more general, so long as there was no contravening evidence. Now it develops that the two attitudes are equally general, and any (positive) metric on a manifold can be realized by an appropriate imbedding in Euclidean space.



John F. Nash



Nicolaas Kuiper

Nash-Kuiper theorem, 1954-55

If $h : (M^n, g) \to \mathbb{E}^k$ with $k > n$ is a short embedding ($h^*\langle \cdot, \cdot \rangle_{\mathbb{E}^k} < g$), then $\forall \varepsilon > 0$ there exists a C^1 isometric $f : (M^n, g) \to \mathbb{E}^k$ s.t.

$$\|f - h\|_{C^0} < \varepsilon$$

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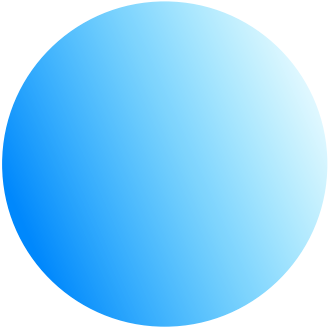

$\xrightarrow{\times(\varepsilon/2)}$

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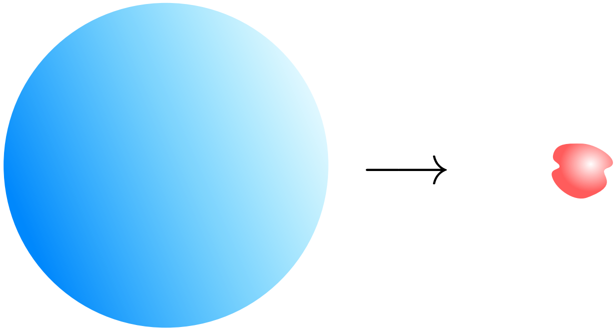

Nash's Method in a Nutshell

- Let h s.t. $g - h^*\langle \cdot, \cdot \rangle_{\mathbb{E}^k}$ is a metric, i.e. h is short.

Nash's Method in a Nutshell

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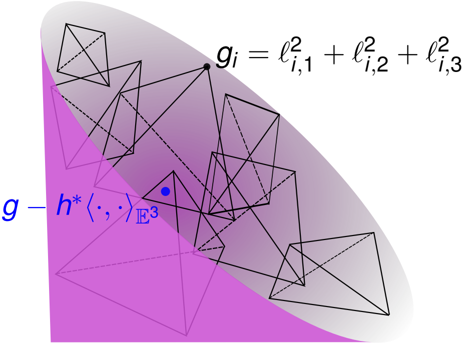
- Choose a locally finite cover of Sym_n^+ by simplices.



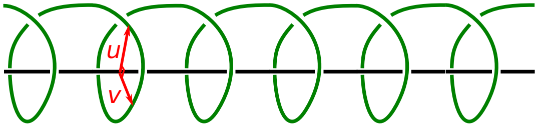
$$g(p) - h^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k}(p) = \sum_\sigma \varphi_\sigma(p) \sum_{i\in\sigma} \alpha_i(p) g_i = \sum_{i,j} a_{i,j}(p)\ell_{i,j}^2$$

Nash's Method in a Nutshell

$$\boxed{g(p) - h^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k}(p) = \sum_{i,j} a_{i,j}(p)\ell_{i,j}^2}$$

- Step i, j: Replace h by

$$h_{i,j} = h + \frac{\sqrt{a_{i,j}}}{N_{i,j}}\left(\cos(N_{i,j}\ell_{i,j})u + \sin(N_{i,j}\ell_{i,j})v\right)$$



$$h_{i,j}{}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k} - h^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k} = a_{i,j}(p)\ell_{i,j}^2 + O(1/N_{i,j})$$

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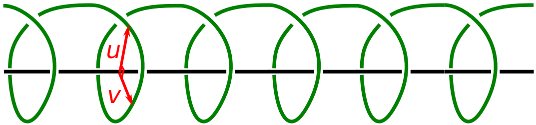

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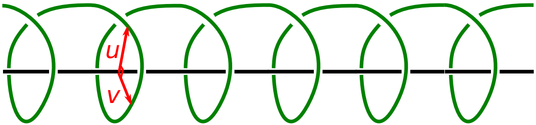

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- Stage = all steps $i, j \rightsquigarrow h_1$.

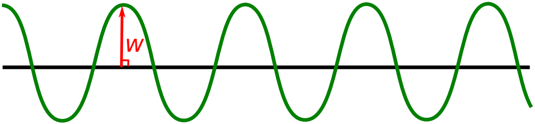
Repeating the stages we get $h_1, h_2, \ldots$. Choosing the $N_{i,j}$ large enough $h_k \xrightarrow{C^1} h_\infty$ with h_∞ a C^1 isometric embedding.

Nash's Method Revised by Kuiper

$$\boxed{g(p) - h^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k}(p) = \sum_{i,j} a_{i,j}(p)\ell_{i,j}^2}$$

- Step i, j: In a suitable chart, replace h by

$$h_{i,j} = h - \frac{a_{i,j}}{4N_{i,j}}\sin(2N_{i,j}\ell_{i,j})\frac{\partial f}{\partial x} + \frac{\sqrt{a_{i,j}}}{N_{i,j}}\sin\left(N_{i,j}\ell_{i,j} - \frac{a_{i,j}}{4}\sin(2N_{i,j}\ell_{i,j})\right)w$$



$$h_{i,j}{}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k} - h^*\langle\cdot,\cdot\rangle_{\mathbb{E}^k} = a_{i,j}(p)\ell_{i,j}^2 + \text{small terms}$$

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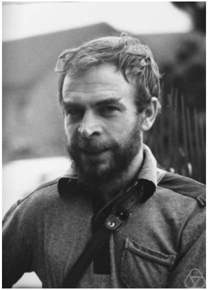
Can you guess the shape of h_∞?

From PDEs to Differential Relations

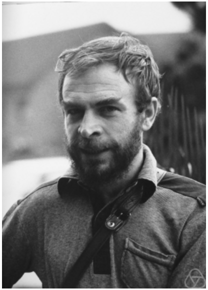


Misha Gromov

$f : (S, g) \to \mathbb{E}^3$ is an isometry if

$$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \rangle_{\mathbb{E}^3} = E, \quad \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_{\mathbb{E}^3} = F, \quad \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \rangle_{\mathbb{E}^3} = G$$

From PDEs to Differential Relations



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$\Leftrightarrow j^1 f : p = (x, y) \mapsto (p, f(p), \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p))$ satisfies

$$R(j^1 f) = (0, 0, 0), \quad \text{where}$$

$$R(p, f, u, v) = (\langle u, u \rangle_{\mathbb{E}^3} - E(p), \langle u, v \rangle_{\mathbb{E}^3} - F(p), \langle v, v \rangle_{\mathbb{E}^3} - G(p))$$

From PDEs to Differential Relations

Idea: Decouple the derivatives from the map to solve

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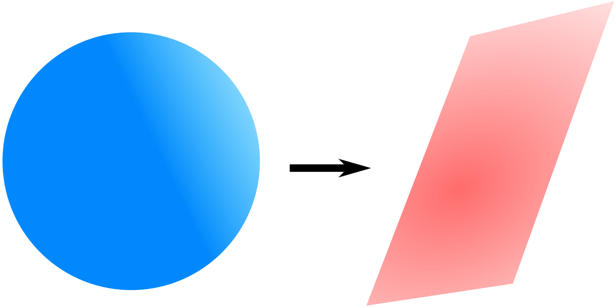
A (counter-)example: There is no immersion $\mathbb{S}^2 \to \mathbb{R}^2$.



$$\mathcal{I} := \{(p, f, L) \mid p \in \mathbb{S}^2, f \in \mathbb{R}^2, L \in \mathcal{L}(T_p\mathbb{S}^2, T_f\mathbb{R}^2) : \quad \text{rank } L = 2\}$$

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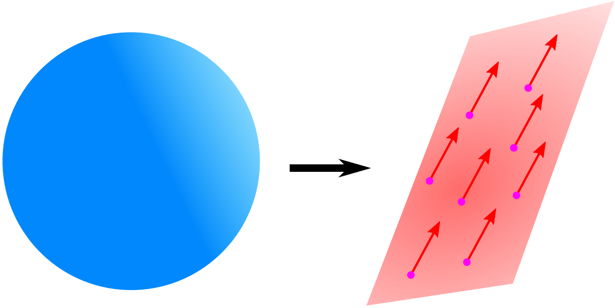

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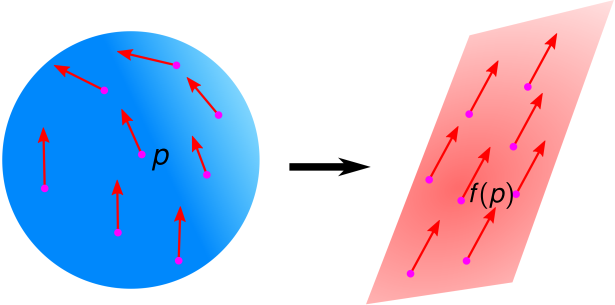

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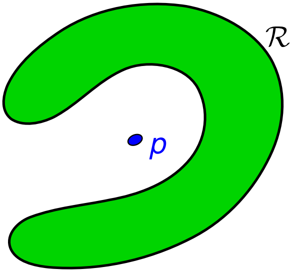
1-dimensional Convex Integration

Convex integration is a tool invented by Gromov to prove the h-principle for many differential relations.



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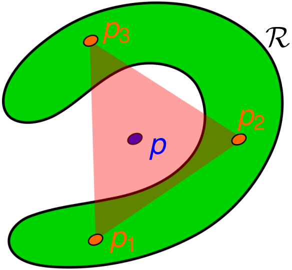

$$p = \sum \alpha_i p_i$$

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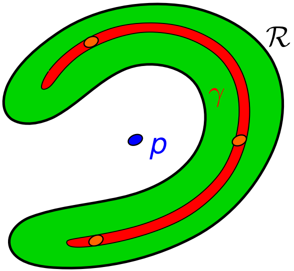

$$p = \int_I \gamma, \quad \gamma \subset \mathcal{R}$$

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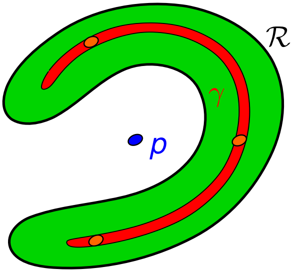


$$p = \int_{\mathbb{S}^1} \gamma, \quad \gamma \subset \mathcal{R}$$

A simple observation

1-dimensional Convex Integration

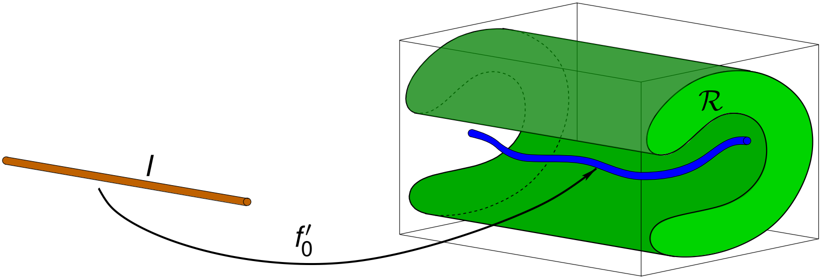
Lemma (Gromov, 1973)

Let $f_0 : I \to \mathbb{R}^3$. For all $x \in I$, suppose $\mathcal{R}_x \subset \mathbb{R}^3$ is open and

$$f_0'(x) \in IntConv(\mathcal{R}_x).$$

Then, $\forall \varepsilon > 0$, there exists a true solution f of $\mathcal{R} = \cup_x \mathcal{R}_x$ s.t.

$$\|f - f_0\|_{C^0} < \varepsilon$$



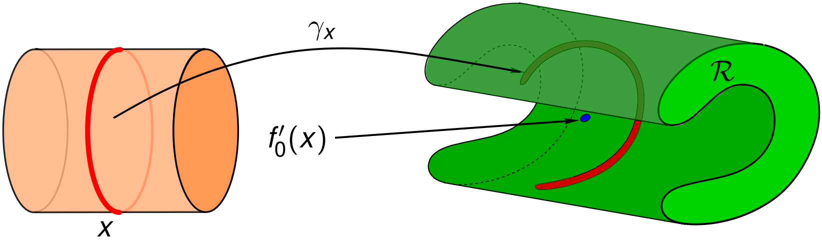
1-dimensional Convex Integration

Step 1
Build a continuous family of loops

$$\begin{array}{rccc} \gamma : & I \times \mathbb{S}^1 & \to & \color{green}{\mathcal{R}} \\ & (x, s) & \mapsto & \gamma_x(s) \end{array}$$

such that

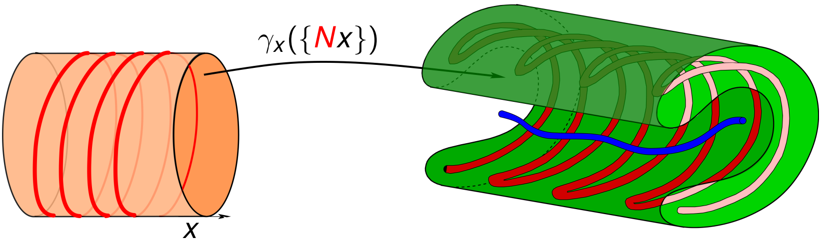
$$\forall x \in I, \quad f_0'(x) = \int_{\mathbb{S}^1} \gamma_x$$



Step 2
Put

$$\boxed{f(x) := f_0(0) + \int_0^x \gamma_s(\{Ns\})ds}$$

where $N \in \mathbb{N}^*$ et $\{Ns\}$ is the fractional part of Ns.



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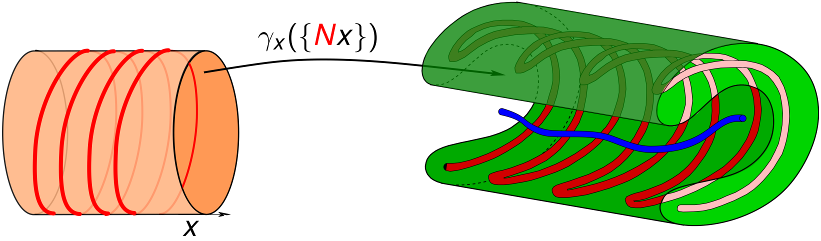

$f'(x) = \gamma_x(\{Nx\}) \in \mathcal{R}_x$ and

$$f(x) \approx f_0(0) + \sum_{i=0}^{\lfloor Nx \rfloor} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \gamma_s(\{Ns\})ds \approx f_0(0) + \sum_{i=0}^{\lfloor Nx \rfloor} \frac{1}{N} f_0'(\frac{i}{N}) \approx f_0(x)$$

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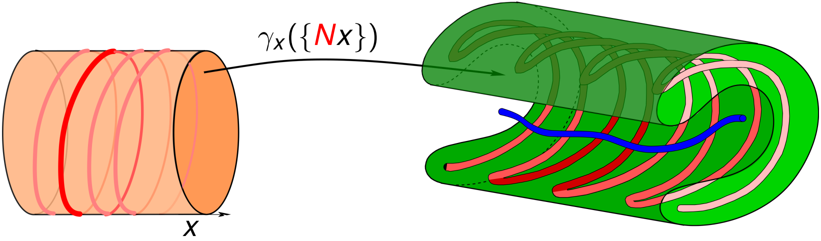

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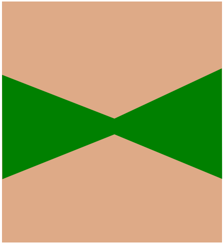
The h-Principle for Ample Relations

Theorem (Gromov, 1973)

Let $\mathcal{R} \subset J^1(M, N)$ be an open and ample differential relation. Then the inclusion of true solutions into the space of formal solutions is a weak homotopy equivalence.



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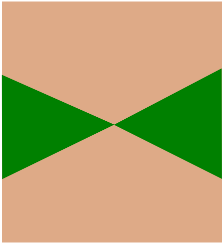

non-ample

The relation of immersions

The differential relation of immersions from M^m to N^n satisfies the h-principle for $n > m$. In particular, $\mathbb{S}^2$ can be everted in $\mathbb{R}^3$.

Back to Isometries

The relation $\mathcal{R}_{iso} := \{R = (0,0,0)\}$ of isometries $\mathbb{S}^2 \to \mathbb{E}^3$ is **neither open nor ample** and $\mathbb{S}^2$ is **2-dimensional.**

$$R(p,q,u,v) = (\langle u,u\rangle_{\mathbb{E}^3} - E(p), \langle u,v\rangle_{\mathbb{E}^3} - F(p), \langle v,v\rangle_{\mathbb{E}^3} - G(p))$$

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5. Extend 1D-CI to 2D.

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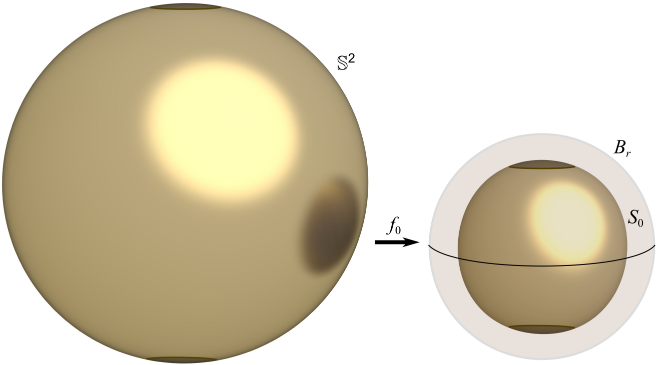

Back to Isometries

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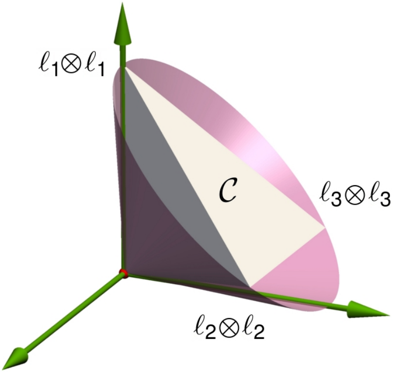


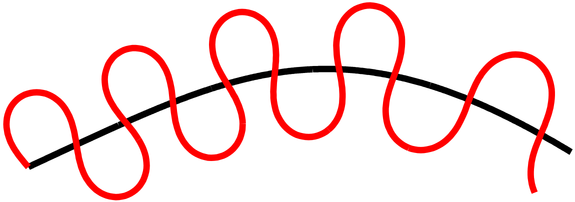
$$g - f_0^* \langle \cdot, \cdot \rangle_{\mathbb{E}_3} = \sum_{i=1}^{3} \rho_i \ell_i \otimes \ell_i$$

$$g_i = f_{i-1}^* \langle \cdot, \cdot \rangle_{\mathbb{E}_3} + \rho_i \ell_i \otimes \ell_i$$

Back to Isometries

- Extend 1D-CI to 2D.

$$\gamma_s(Ns) = r(s)\mathbf{e}^{i\alpha_s \cos(2\pi Ns)}$$



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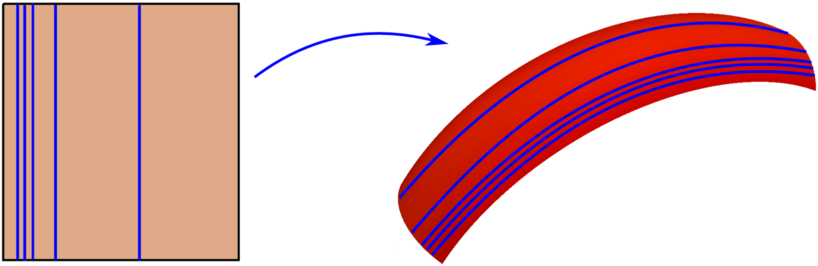

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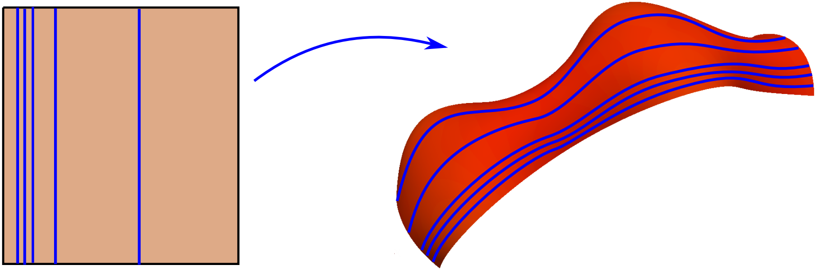

Corrugations

Back to Isometries

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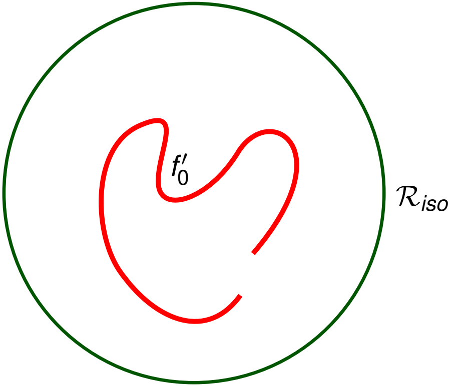

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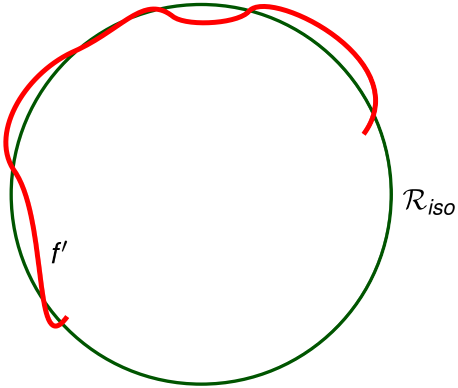

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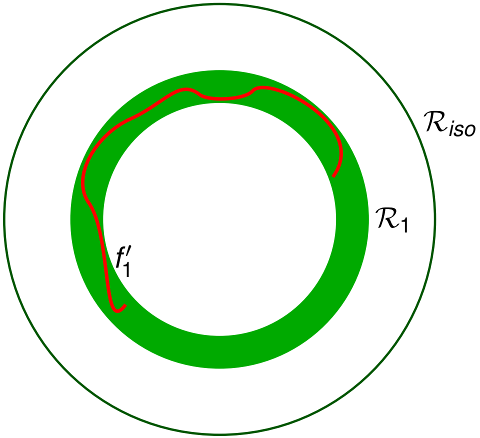

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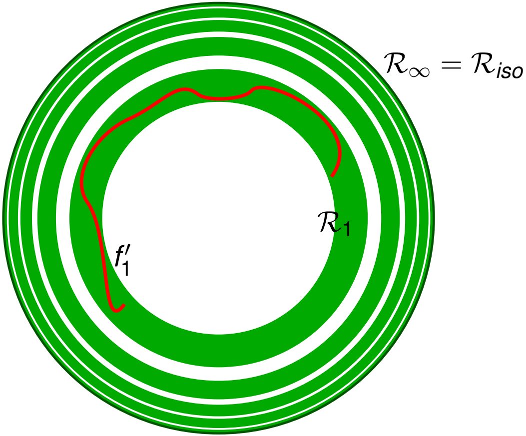

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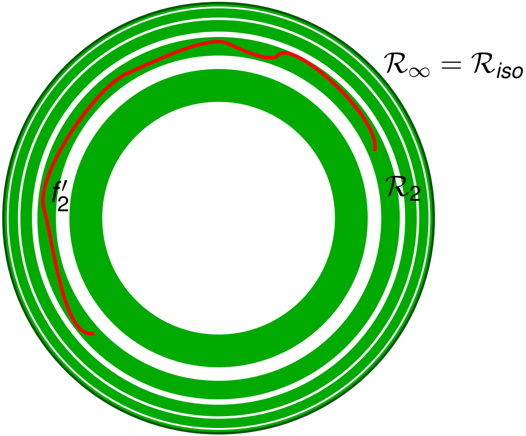

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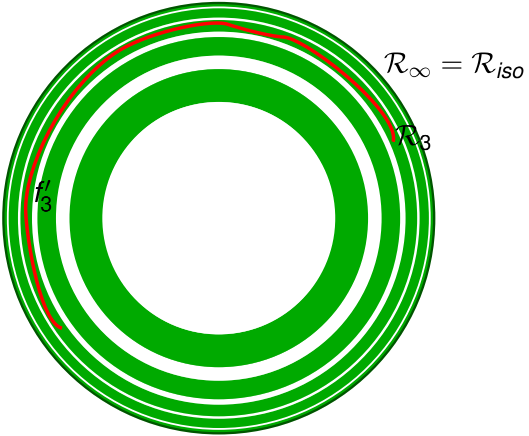
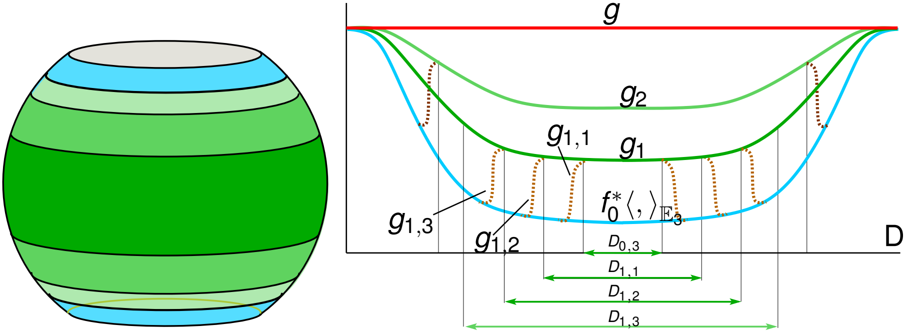


Back to Isometries

- Deal with boundary conditions.



Compute embeddings $f_{i,j}$ so that $f_{i,j}^*\langle , \rangle_{\mathbb{E}^3} \approx g_{i,j}$.

Back to Isometries

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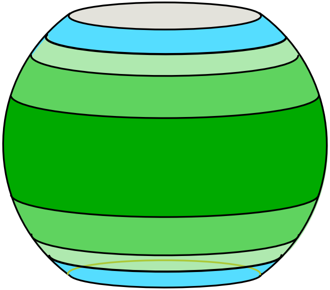
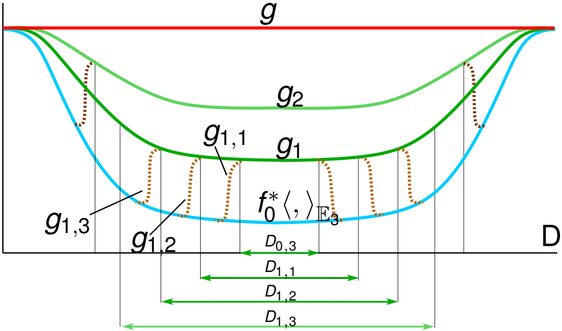
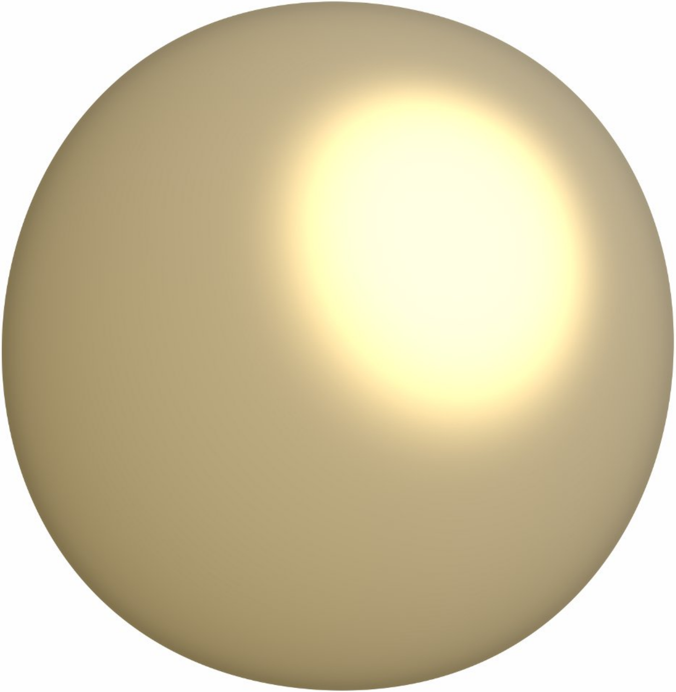
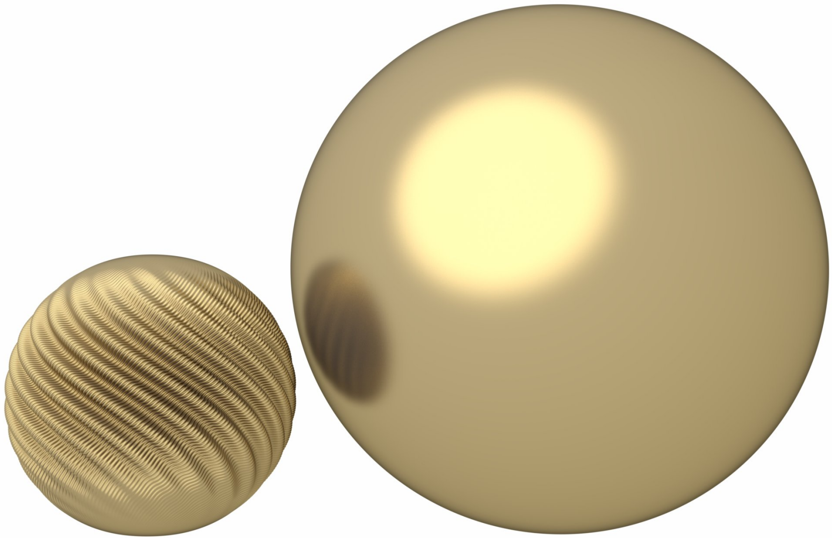


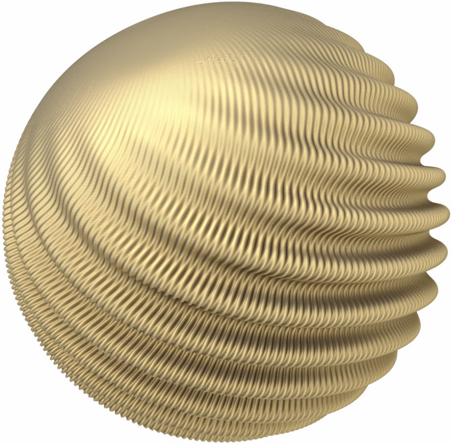


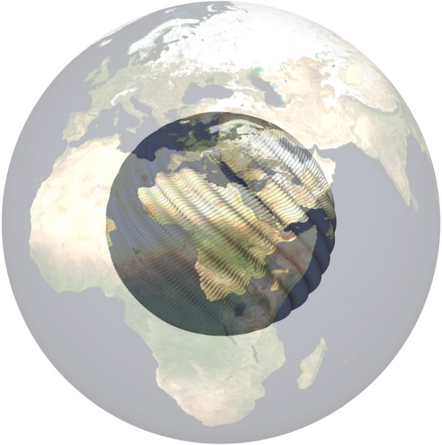
Compute embeddings $f_{i,j}$ so that $f_{i,j}^*\langle,\rangle_{\mathbb{E}^3} \approx g_{i,j}$.

$\Longrightarrow$ **sequence** $f_0, f_{1,1}, f_{1,2}, f_{1,3}, f_{2,1}, \ldots$
C^1 **converging to an isometry.**

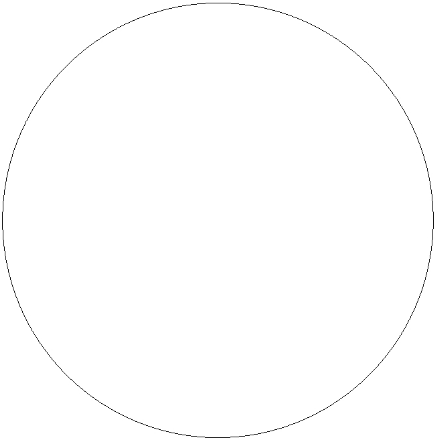








The IC process applied on a circle of radius < 1 .



The C^1 Fractal Structure

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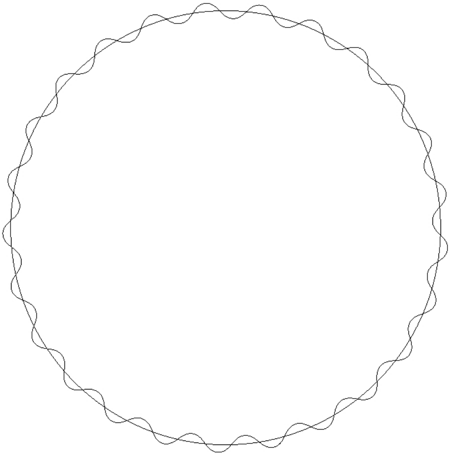

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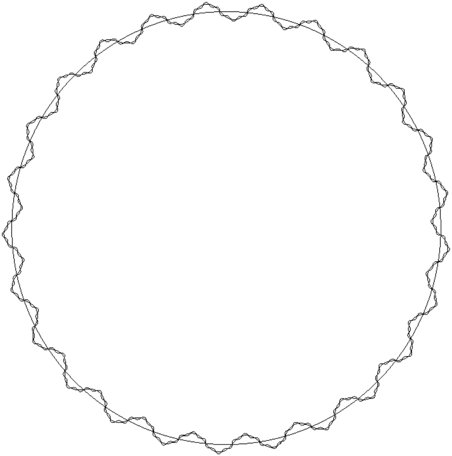

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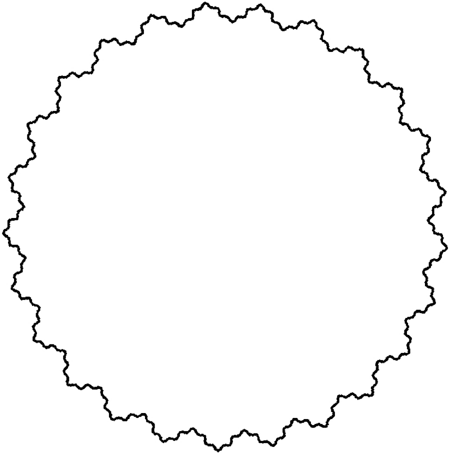
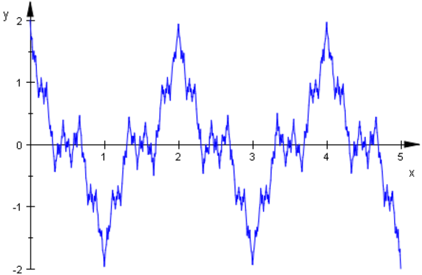


The C^1 Fractal Structure

The IC process applied on a circle of radius < 1 .

$$\forall x \in \mathbb{S}^1, \quad \mathbf{n}_\infty(x) = \left(\prod_{k=1}^{\infty} e^{i\alpha_k(x)\cos 2\pi N_k x} \right) \mathbf{n}_0(x) = e^{iA_\infty(x)} \mathbf{n}_0(x)$$

where $\mathbf{n}_0$ is $\perp$ to f_0 and $A_\infty(x) = \sum_{k=1}^{\infty} \alpha_k(x) \cos 2\pi N_k x$.



$$\dim_{\mathcal{H}} graph(\sum_{k=0}^{\infty} a^k \cos(2\pi b^k x)) \leq \ln(a)/\ln(b) + 2$$

The IC process applied on a circle of radius < 1 .

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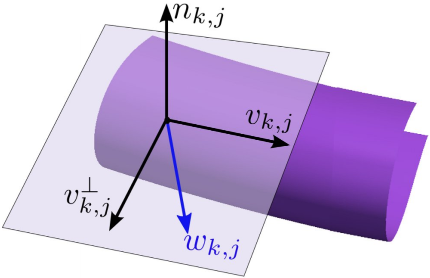
where $\mathbf{n}_0$ is $\perp$ to f_0 and $A_\infty(x) = \sum_{k=1}^{\infty} \alpha_k(x)\cos 2\pi N_k x$.

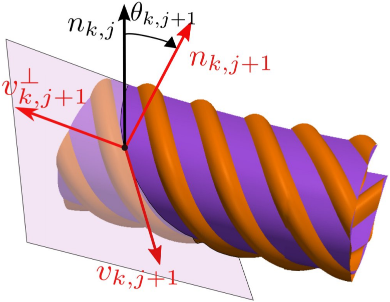
$$\begin{pmatrix} \mathbf{t}_\infty \\ \mathbf{n}_\infty \end{pmatrix} = \left(\prod_{k=0}^{\infty} \mathcal{C}_k\right)\begin{pmatrix} \mathbf{t}_0 \\ \mathbf{n}_0 \end{pmatrix}$$

where $\mathcal{C}_k = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix}$ and $\theta_k(x) = \alpha_k(x)\cos 2\pi N_k x$

Let

$$(\mathbf{v}^{\perp}_{k,j+1}\ \mathbf{v}_{k,j+1}\ \mathbf{n}_{k,j+1})^t = \mathcal{C}_{k,j+1} \cdot (\mathbf{v}^{\perp}_{k,j}\ \mathbf{v}_{k,j}\ \mathbf{n}_{k,j})^t, \quad \mathcal{C}_{k,j+1} \in SO(3)$$





The **corrugation matrix** is:

$$R(k,i) = \prod_{\ell=k+1}^{\infty} \left(\prod_{j=1}^{3} \mathcal{C}_{\ell,j} \right) \prod_{j=i}^{3} \mathcal{C}_{k,j}$$

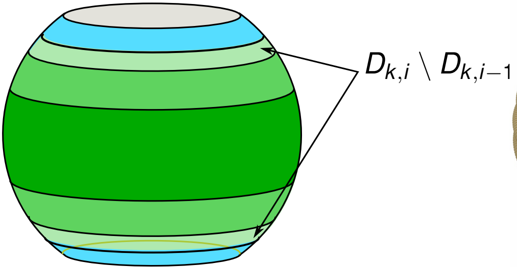
The C^1 Fractal Structure

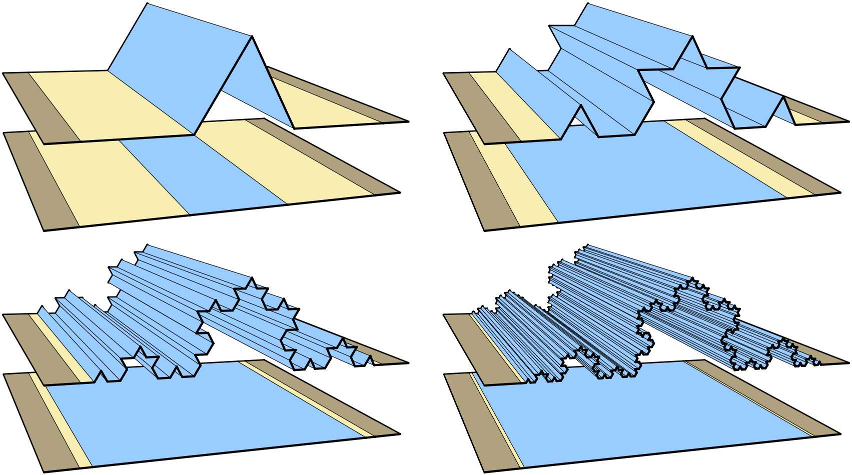
The **corrugation matrix**: $R(k,i) = \prod_{\ell=k+1}^{\infty} \left(\prod_{j=1}^{3} \mathcal{C}_{\ell,j}\right) \prod_{j=i}^{3} \mathcal{C}_{k,j}$.

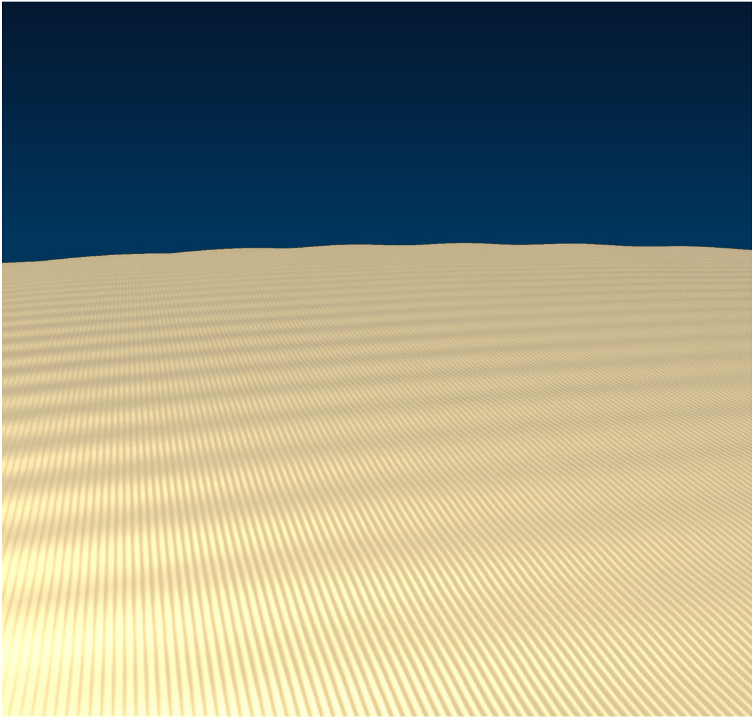
Theorem (C^1 fractal expansion)

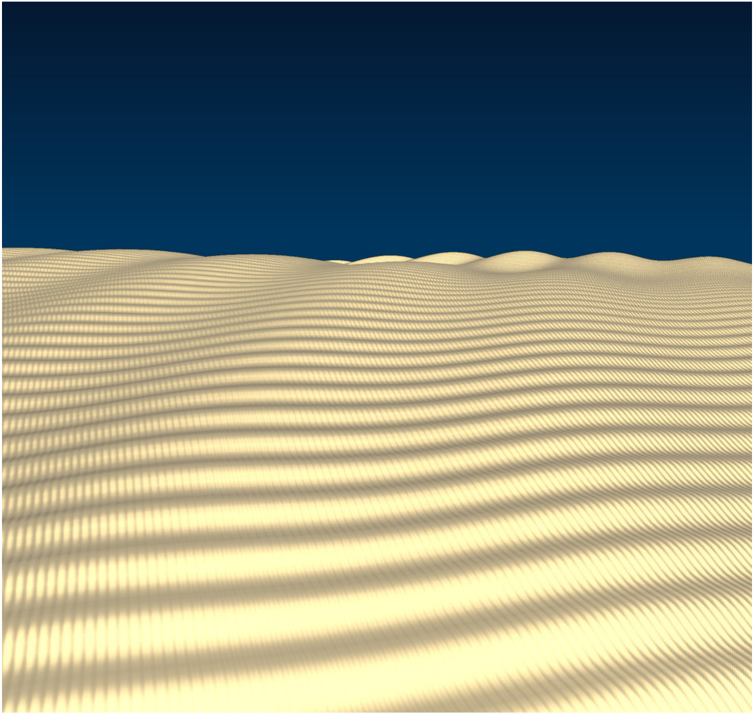
The Gauss map $\mathbf{n}_\infty$ of $f_\infty := \lim_{k\to+\infty} f_{k,3}$ over $D_{k,i} \setminus D_{k,i-1}$, where $k \geq 1$ and $i \in \{1,2,3\}$, is given by

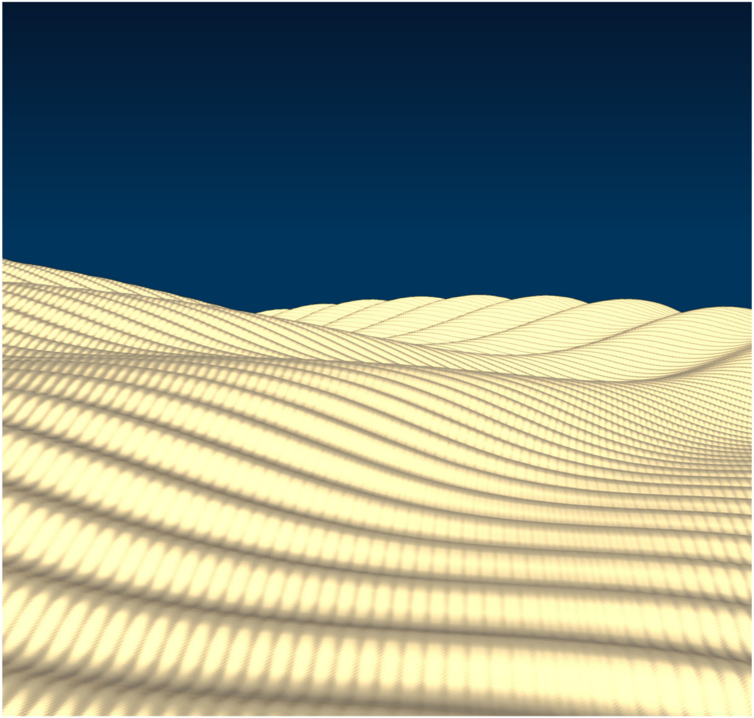
$$\mathbf{n}_\infty^t = (0 \quad 0 \quad 1) \cdot R(k,i) \cdot (\mathbf{v}_{0,i}^\perp \quad \mathbf{v}_{0,i} \quad \mathbf{n}_0)^t$$

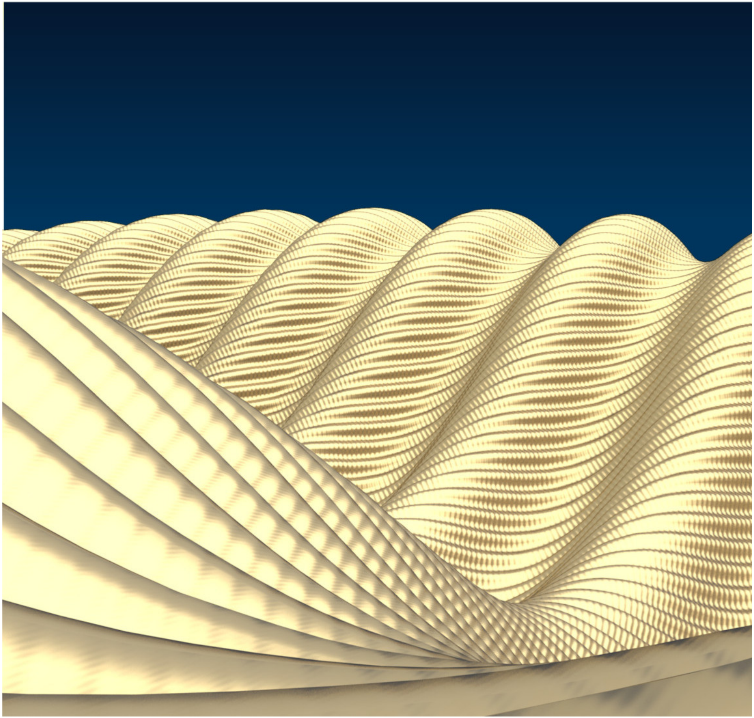


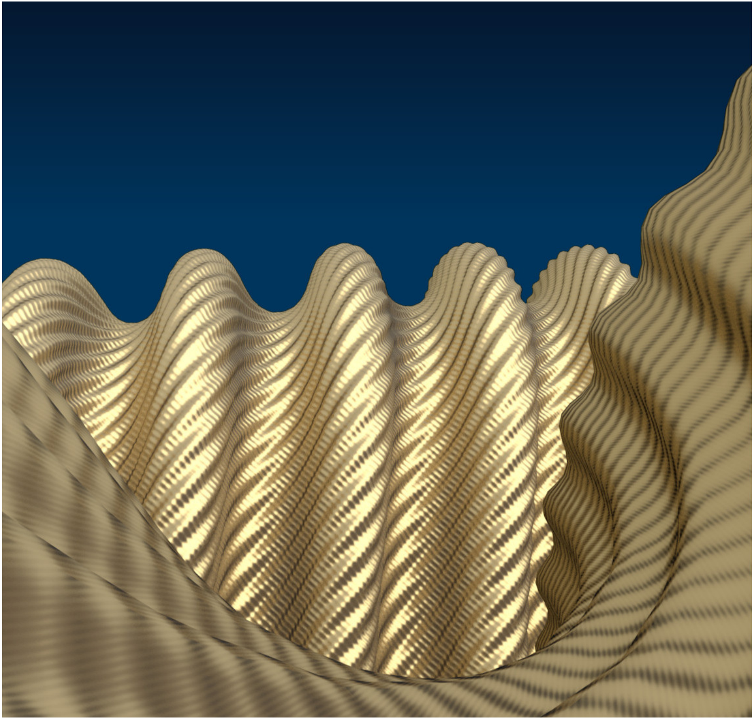


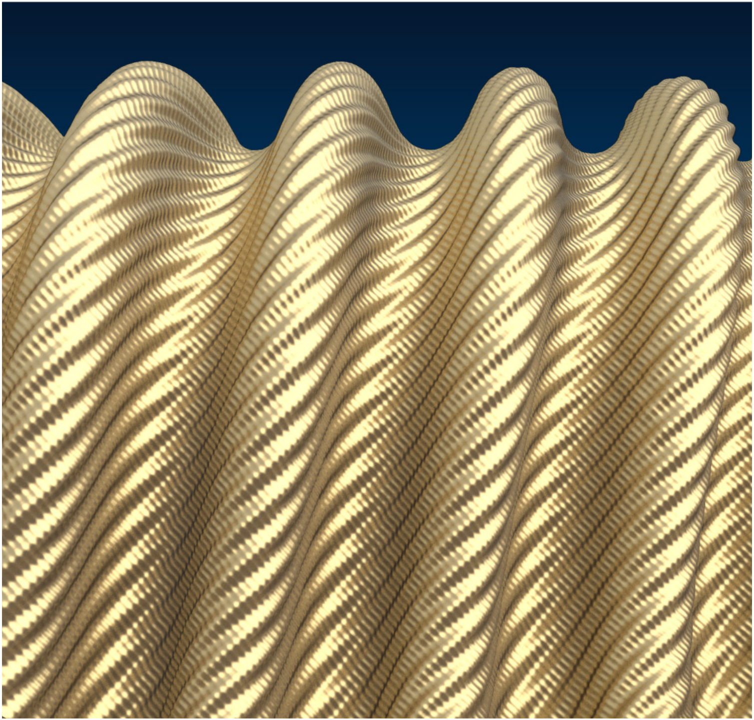


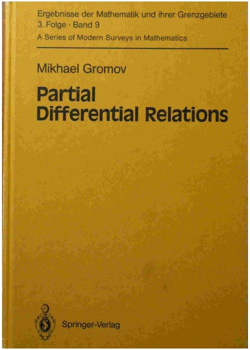








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A Series of Modern Surveys in Mathematics
Mikhael Gromov
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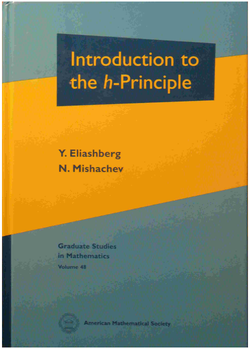
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Y. Eliashberg
N. Mishachev
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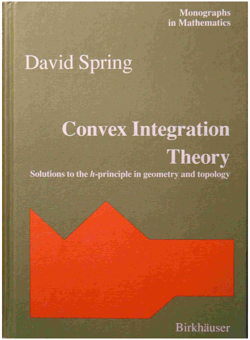
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Convex Integration Theory
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The HEVEA Project

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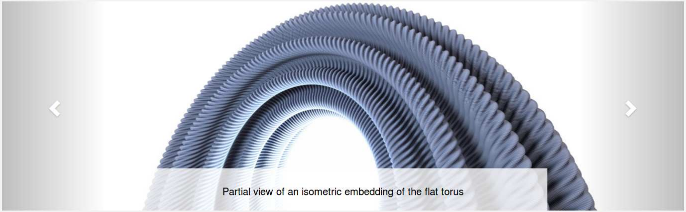

Partial view of an isometric embedding of the flat torus

The flat torus

The reduced sphere

Ongoing

Website designed by Mélanie Theillière

Thank you!