Hard Problems in 3-Manifold Topology

School on Low-Dimensional Geometry and Topology: Discrete and Algorithmic Aspects

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Some NP-Hard Problems in 3-Manifold Topology

Jones Polynomial (#P-hard) - Jaeger, Vertigan, Welsh - 1990

Witten, Reshetikhin, Turaev Invariant τ_4 (#P-hard) - Kirby, Melvin - 2004

3-Manifold Knot Genus - Agol, Hass, Thurston - 2006

Taut Angle Structure - Burton, Spreer - 2013

Turaev-Viro invariants (#P-hard) - Burton, Maria, Spreer - 2015

Immersibility - Burton, Colin de Verdière, de Mesmay - 2016

Sublink, Upper Bound for the Thurston complexity of an unoriented classical link - Lackenby - 2016

Heegaard Genus - Bachman, Derby-Talbot, Sedgwick - 2016

Non Orientable Surface Embeddability - Burton, de Mesmay, Wagner - 2017

Embed$_{2\to3}$, Embed$_{3\to3}$, 3-Manifold Embeds in S^3 - de Mesmay, Rieck, Sedgwick, Tancer - 2017

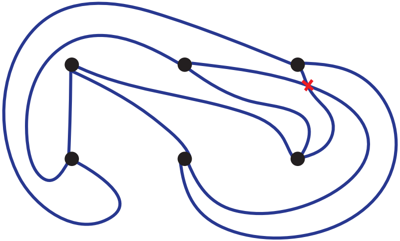
Trivial Sub-Link, Unlinking Number, Reidemeister Distance/Defect, 4-Ball Euler Char 0 - de Mesmay, Rieck, Sedgwick, Tancer - 2018

Embeddings in $\mathbb{R}^d$

$\textsc{Embed}_{k \to d}$

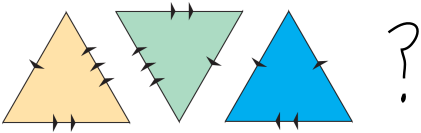
Problem: $\textsc{Embed}_{k \to d}$

Given a k-dimensional simplicial complex, does it admit a piecewise linear embedding in $\mathbb{R}^d$?

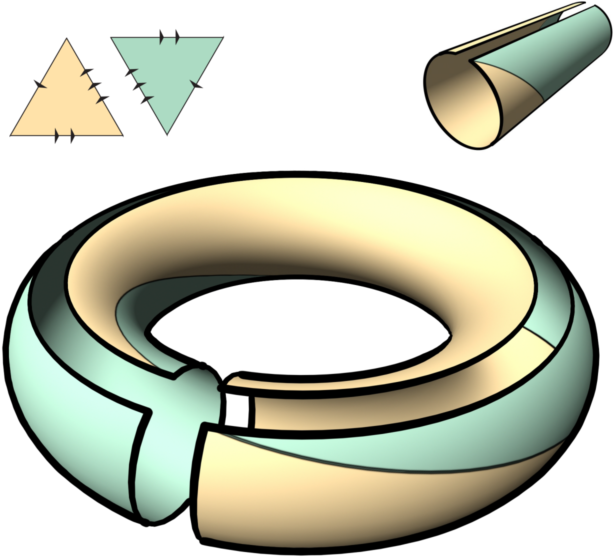


$\textsc{Embed}_{1 \to 2}$ is Graph Planarity

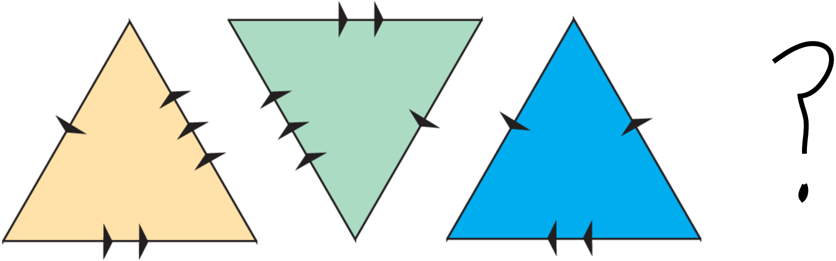
$\textsc{Embed}_{2 \to 3}$: does this 2-complex embed in $\mathbb{R}^3$?



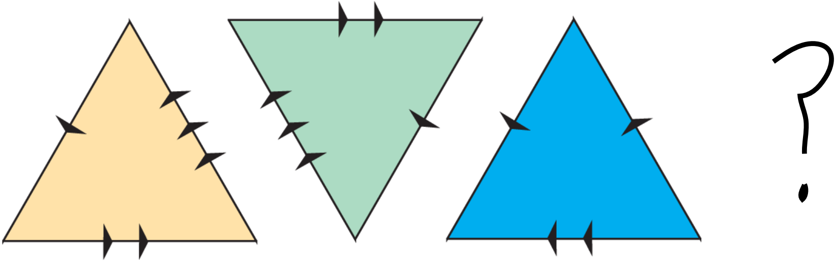
Does it embed?



Does it embed?

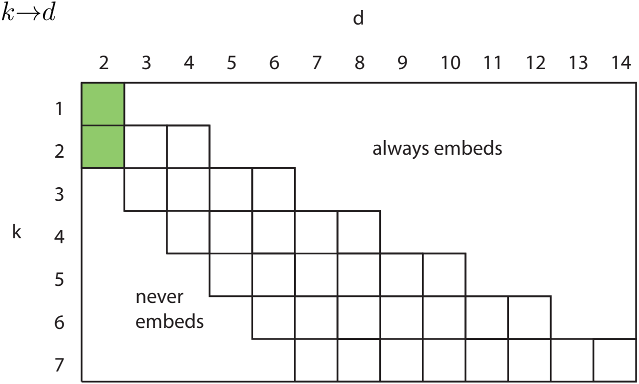


Does it embed?



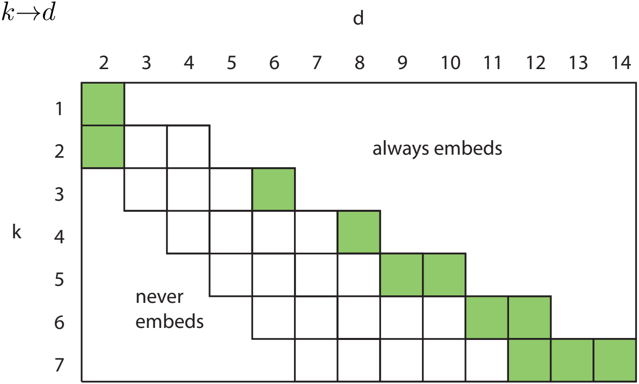
Yes, but must change the embedding of yellow/green torus from the previous picture.

EMBED$_{k \to d}$



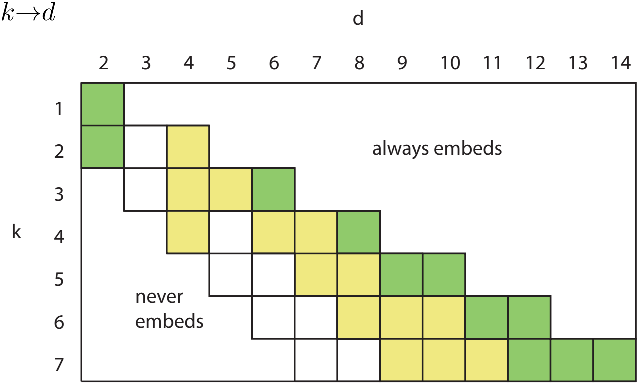
Polynomially decidable - Hopcroft, Tarjan 1971

$\textsc{Embed}_{k \to d}$



- Polynomially decidable - Hopcroft, Tarjan 1971 ; Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner 2013-2017

EMBED$_{k \to d}$



- Polynomially decidable - Hopcroft, Tarjan 1971 ; Čadek, Krčál, Matoušek, Sergeraert, Vokřínek, Wagner 2013-2017
- NP-hard - Matoušek, Tancer, Wagner '11

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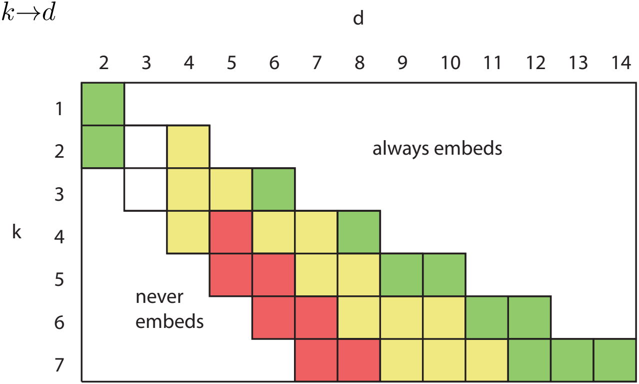

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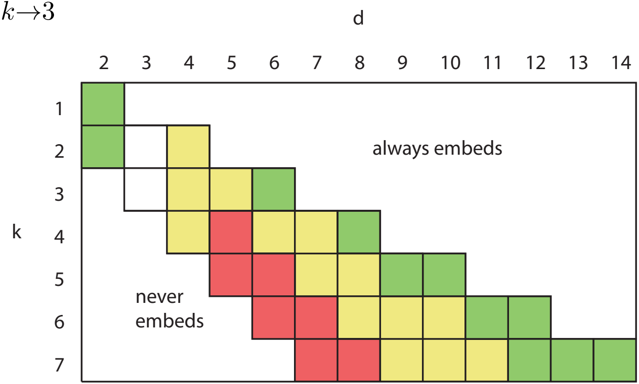
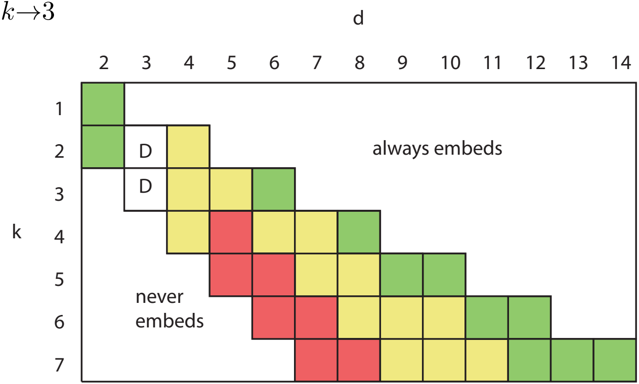
- NP-hard - Matoušek, Tancer, Wagner '11
- Undecidable - Matoušek, Tancer, Wagner '11

$\text{EMBED}_{k\to 3}$



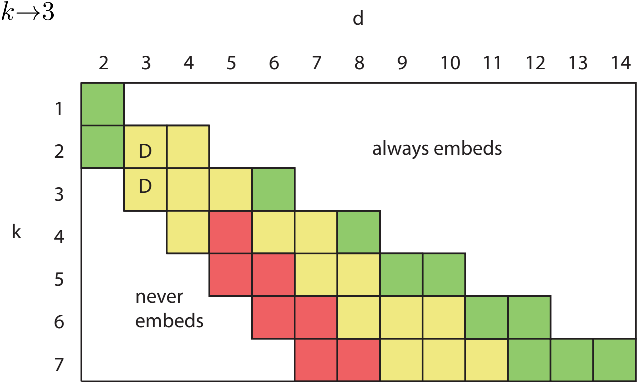
Embed$_{k \to 3}$



Theorem (Matoušek, S', Tancer, Wagner 2014)

The following problems are ***decidable:***
Embed$_{2 \to 3}$,
Embed$_{3 \to 3}$, *and*
3-Manifold Embeds in S^3 (or $\mathbb{R}^3$).

$\text{Embed}_{k\to 3}$



Theorem (de Mesmay, Rieck, S', Tancer 2017)

*The following problems are **NP-hard**:*

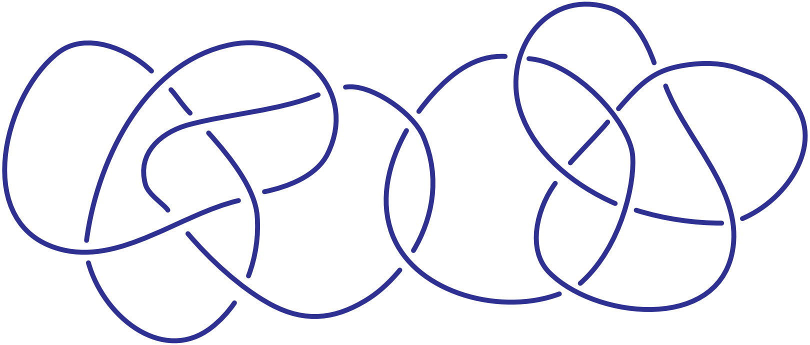
$\text{Embed}_{2\to 3}$,

$\text{Embed}_{3\to 3}$, *and*

3-Manifold Embeds in S^3 (or $\mathbb{R}^3$).

Knots and Links

A link diagram



Reidemeister moves

Reidemeister (1927)

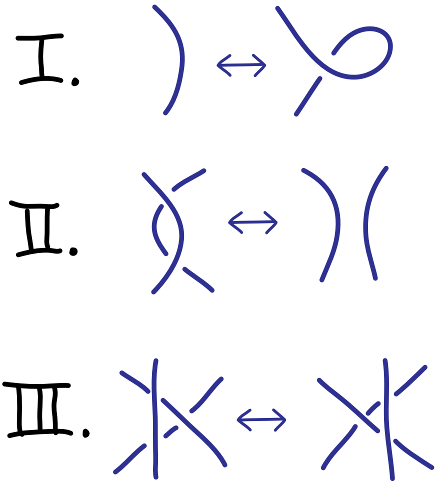
Any two diagrams of a link are related by a sequence of 3 moves (shown to the right).

Question: Reidemeister Distance

How many moves are needed?

Note:

May need to increase number of crossings.



Unlinking Number

Crossing Changes:

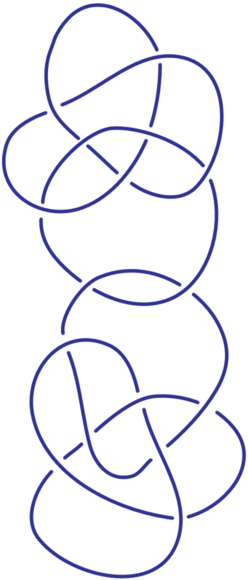
Any link diagram can be made into a diagram of an unlink (trivial) by changing some number of crossings.

Unlinking Number:

The minimum number of crossings *in some diagram* that need to be changed to produce an unlink.

Warning:

Minimum number may not be in the given diagram, so may need Reidemeister moves too.



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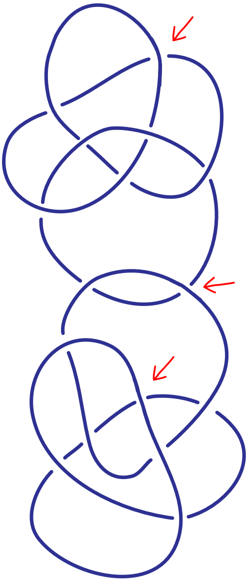


Given a link diagram, 3 Questions:

Triviality

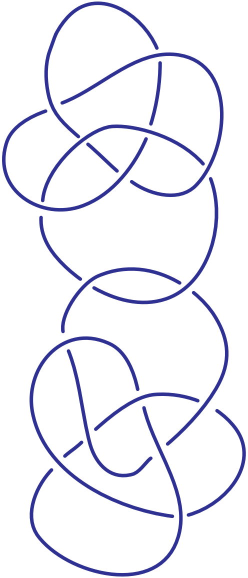
Is it trivial? Can Reidemeister moves produce a diagram with no crossings?

Trivial Sub-link

Does it have a trivial sub-link? How many components?

Unlinking Number

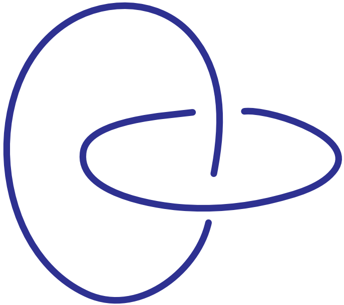
What is the unlinking number? How many crossing changes must be made to produce an unlink?



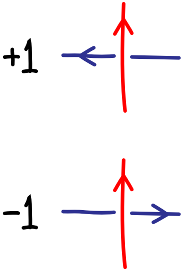
Hopf link

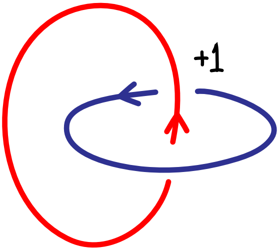
Triviality

Doesn't seem trivial, but how do you prove it?



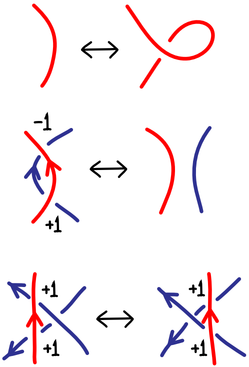
Linking number for two components:

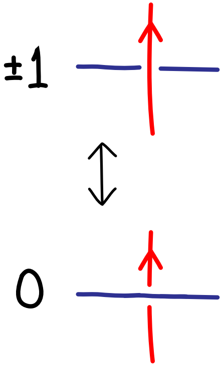




- choose red and blue and orient them
- for crossings of red over blue
- linking number is the sum of $+1$'s and -1's.

Linking number





Reidemeister moves

don't change the linking number!

A crossing change

changes the linking number by ± 1

Hopf Link

Triviality

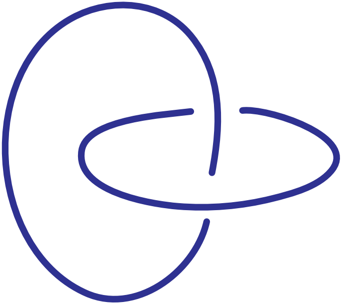
Not trivial. Linking number is not zero.

Trivial Sub-link

Maximal trivial sub-link has one component.

Unlinking Number

Unlinking number 1.



Borromean Rings

Triviality

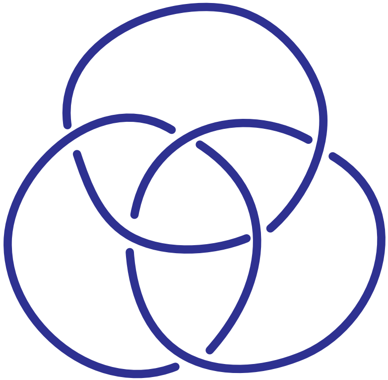
Not trivial. (But harder to prove, linking numbers are 0.)

Trivial Sub-link

Maximal trivial sub-link has **two** components.

Unlinking Number

Unlinking number 2. (Must show that it is greater than 1.)



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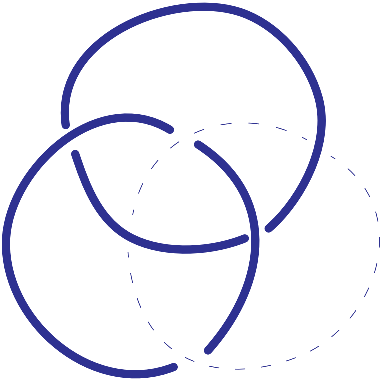

Whitehead Double of the Hopf Link

Triviality

Not trivial. (Requires proof, linking numbers are 0.)

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Maximal trivial sub-link has **one** component.

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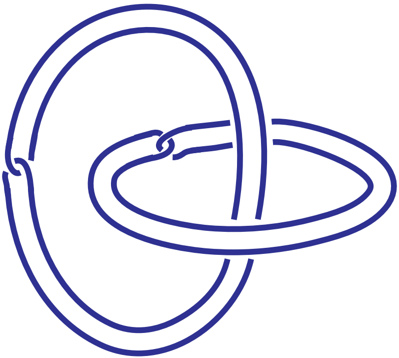
Unlinking number 1.



Whitehead Double of the Borromean Rings

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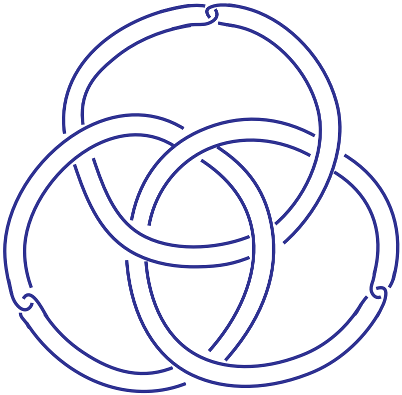


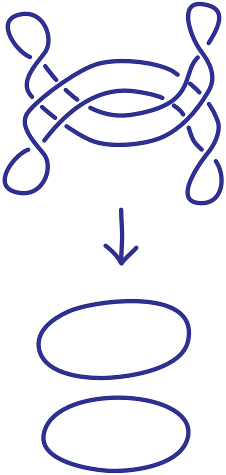
Reidemeister Distance/Defect

Reidemeister Distance

Given two diagrams of the same link, let the *Reidemeister distance* be the number of Reidemeister moves required to get from one to the other.

Special Case: Reidemeister Defect

Given a diagram of a unlink, how many moves are required to remove all crossings? Measure the *defect*, the number of extra moves required:



Reidemeister Distance/Defect

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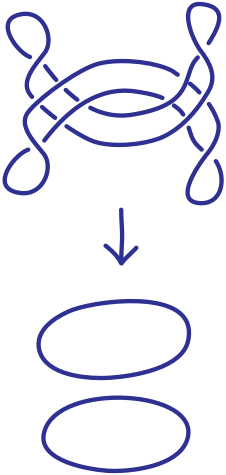
- $\# \; moves \geq 1/2 \; crossings$



Reidemeister Distance/Defect

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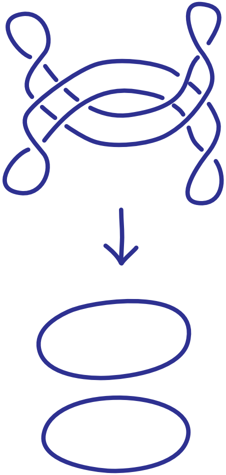
- $\# \; moves \geq 1/2 \; crossings$
- $defect := \# \; moves - 1/2 \; crossings$



Reidemeister Distance/Defect

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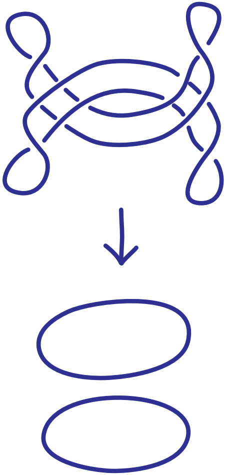
Special Case: Reidemeister Defect

Given a diagram of a unlink, how many moves are required to remove all crossings? Measure the *defect*, the number of extra moves required:

- $\# \, moves \geq 1/2 \; crossings$
- $defect := \# \, moves - 1/2 \; crossings$
- diagram to right: $7 \; moves, \;\; defect = 1$.



Decision Problems for Link Diagrams

Triviality

Given a link diagram, does it represent a trivial link?

Trivial Sub-link

Given a link diagram and a number n, does the link contain a trivial sub-link with n components?

Unlinking Number

Given a link diagram and a number n, can the link be made trivial by changing n crossings (in some diagram(s))?

Reidemeister Defect (for unlink diagrams)

Given a diagram of an unlink and a number n, does the diagram have $defect = n$? I.e., can all crossings be removed with $\frac{1}{2}\ crossings + n$ Reidemeister moves?

What is known?

	NP	NP-hard
TRIVIALITY	✓	unlikely
TRIVIAL SUB-LINK	✓	✓
UNLINKING NUMBER	?	✓
REIDEMEISTER DEFECT	✓	✓
REIDEMEISTER DISTANCE	?	✓

✓ - next slide, ✓ - our results

Reidemeister Defect, Triviality & Trivial Sub-Link are in NP

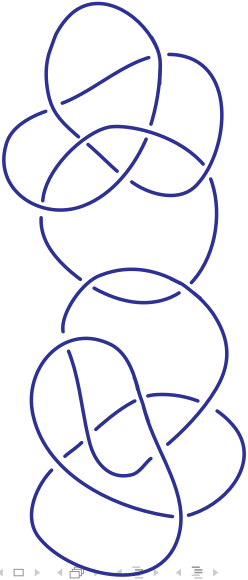
Haken (1961); Hass, Lagarias, and Pippenger (1999)

Unknot recognition is decidable [H], and, in NP [HLP].

Lackenby (2014), (Dynnikov (2006))

For a diagram of an unlink, the number of moves required to eliminate all crossings is bounded polynomially in the number of crossings of the starting diagram.

Thus: Reidemeister Defect, Triviality & Trivial Sub-Link are in NP.



TRIVIAL SUB-LINK is NP-hard

Trivial Sub-link is NP-hard

Problem: Trivial Sub-link

Given a link diagram and a number n, does the link contain a trivial sub-link with n components?

Lackenby (2017)

(Non-trivial) Sub-link is NP-hard.

de Mesmay, Rieck, S' and Tancer (2017)

Trivial Sub-link is NP-hard

Proof is a reduction from 3-SAT:

Given an (exact) 3-CNF formula Φ, there is a link L_Φ that has an n component trivial sub-link if and only if Φ is satisfiable. **($n =$ number of variables)**

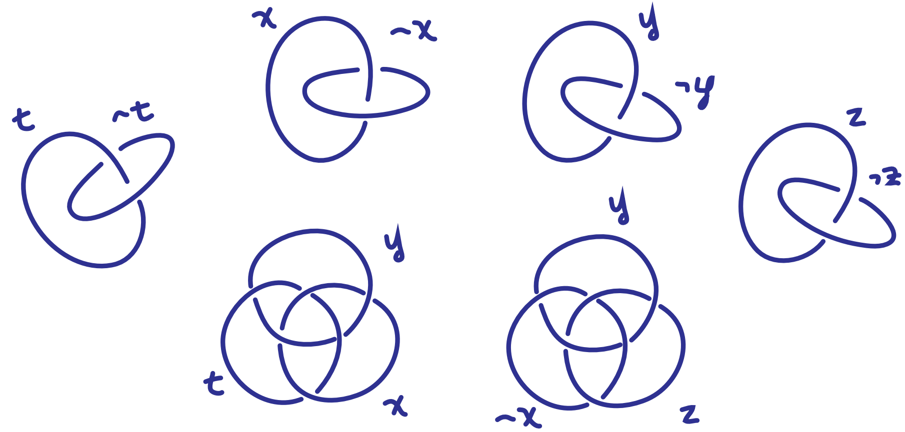
Constructing the link L_Φ :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$

Given an (exact) 3-CNF formula, need to describe a link.

Constructing the link L_Φ :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Draw **Hopf link** for each variable, **Borromean rings** for each clause.

Constructing the link L_Φ :

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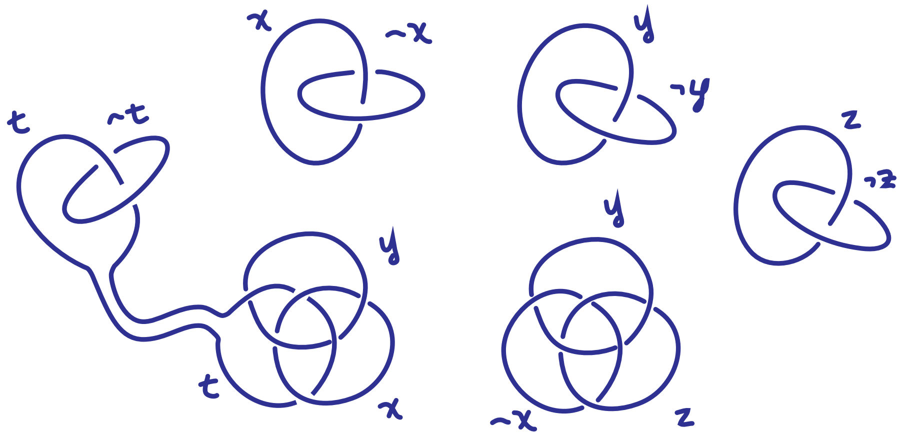

Band each variable to its corresponding variable in the clauses.

Constructing the link L_Φ :

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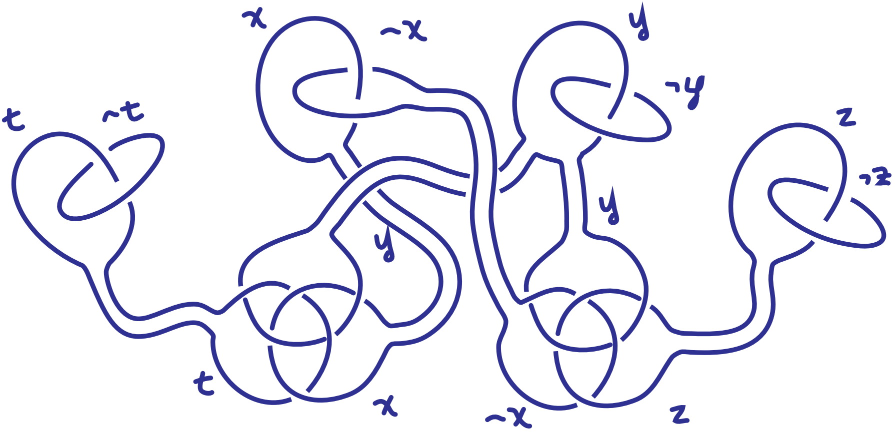

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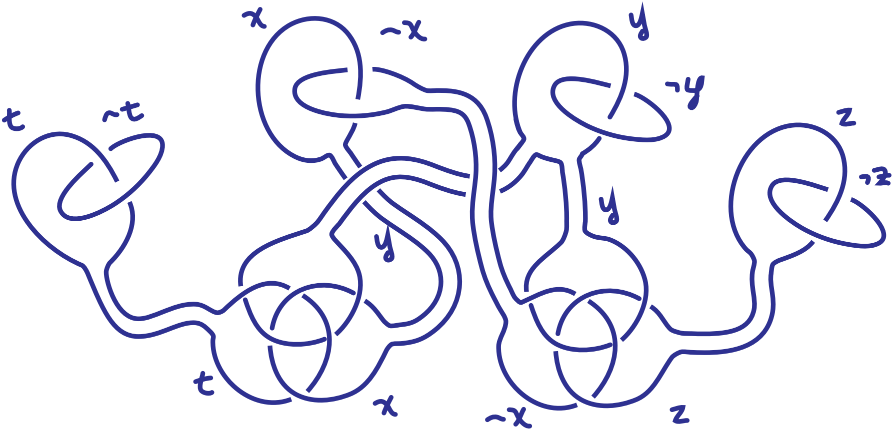
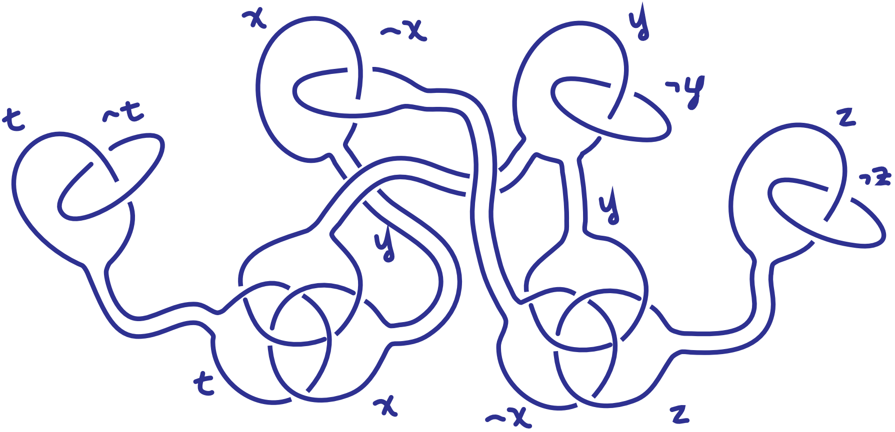


Each component is an unknot.

Φ satisfiable $\implies$ n component trival sub-link

Satisfiable $\implies$ n component trivial sub-link :

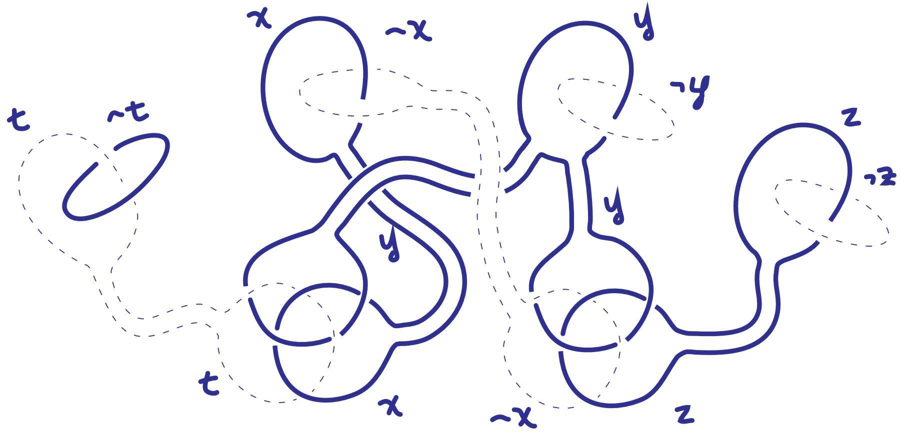
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Satisfiable: $t =$ TRUE; $x, y, z =$ FALSE.

Satisfiable $\implies$ n component trivial sub-link :

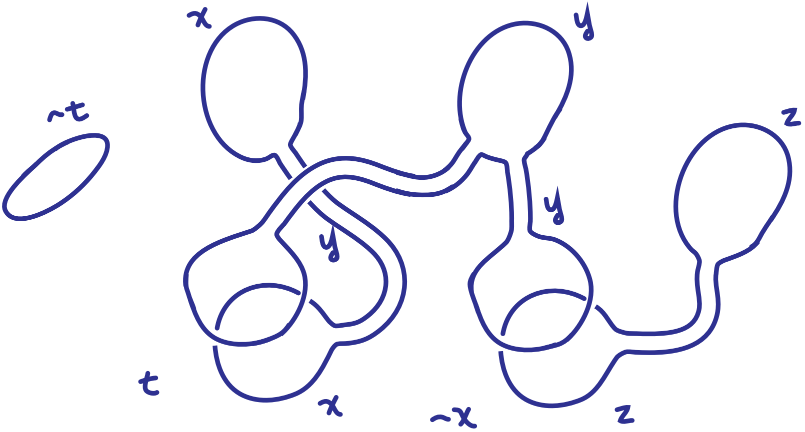
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Erase TRUE components: $t, \neg x, \neg y, \neg z$.

Satisfiable $\implies$ n component trivial sub-link :

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$

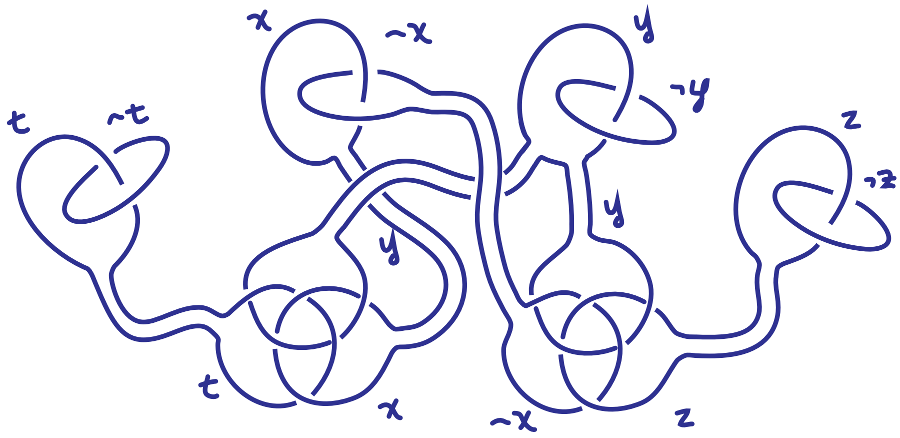


The FALSE components form an n component trivial sub-link.

$$n \text{ component trival sub-link} \implies \Phi \text{ satisfiable}$$

n component trivial sub-link $\implies$ satisfiable:

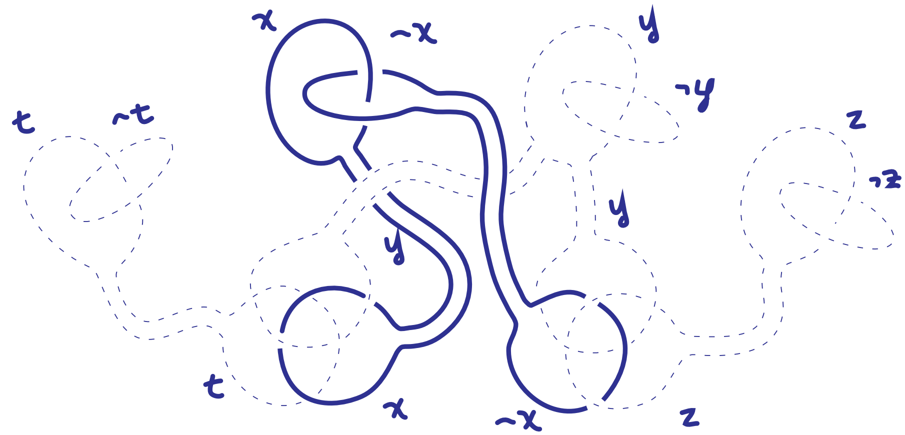
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Label the n trivial link components as FALSE, the others TRUE.

n component trivial sub-link $\implies$ satisfiable:

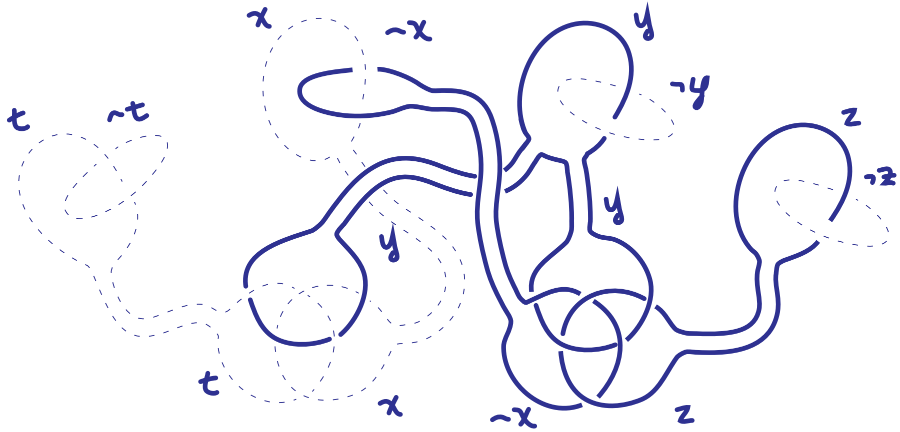
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



For each pair $(x, \neg x)$, one is TRUE the other FALSE.

n component trivial sub-link $\implies$ satisfiable:

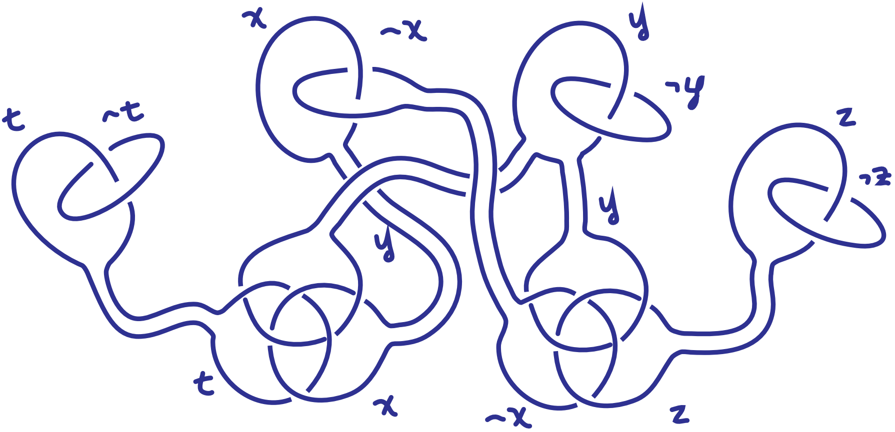
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Each clause has a TRUE. (Borromean rings not sub-link of trivial link.)

n component trivial sub-link $\implies$ satisfiable:

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$

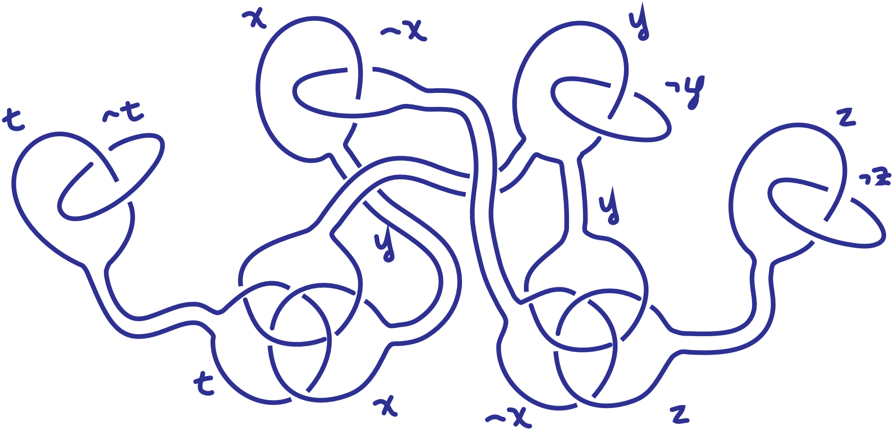


Therefore, Φ is satisfiable.

UNLINKING NUMBER is NP-hard

Unlinking Number is NP-hard

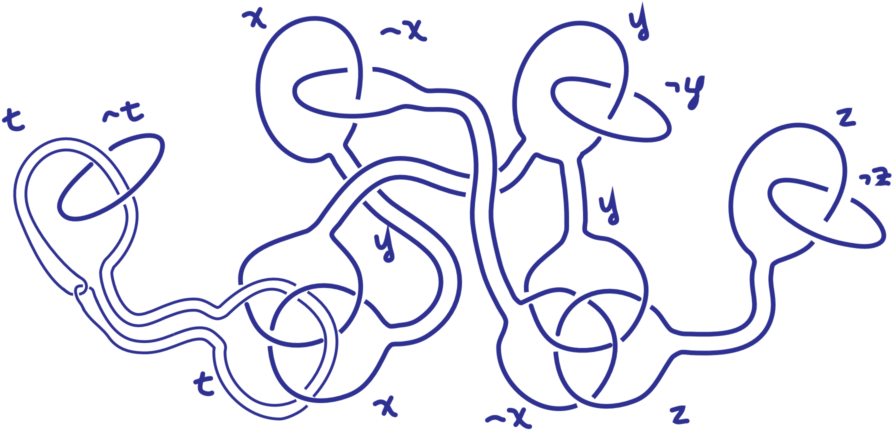
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Related construction.

Unlinking Number is NP-hard

$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



But replace each component with its Whitehead Double!

Unlinking Number is NP-hard

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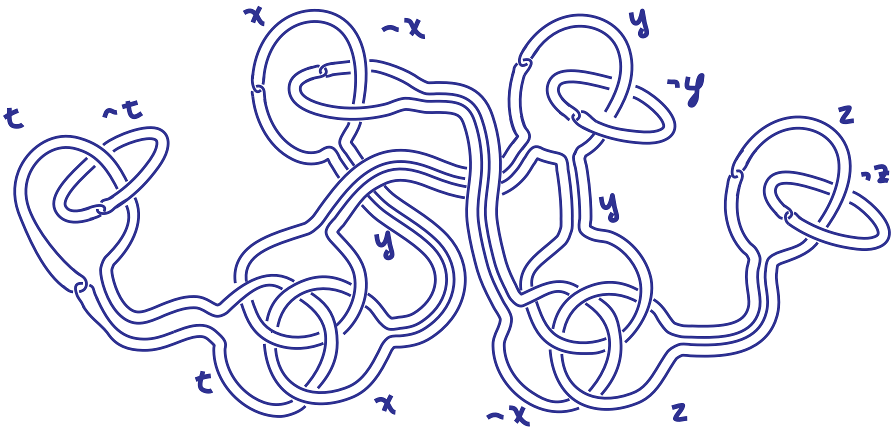

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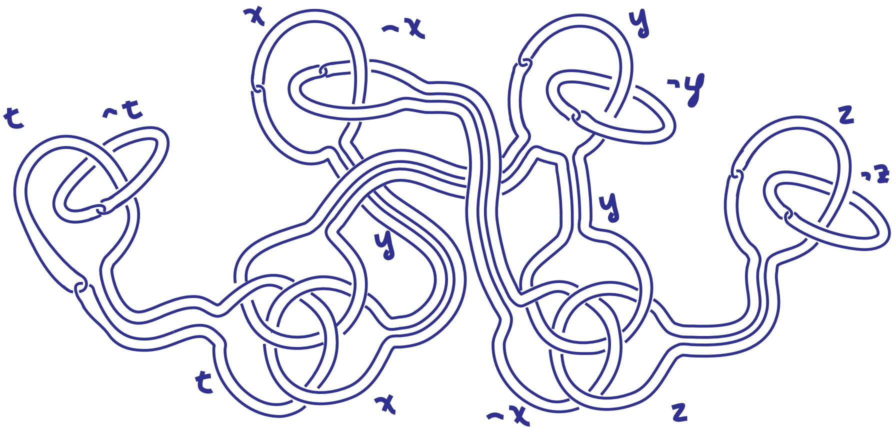
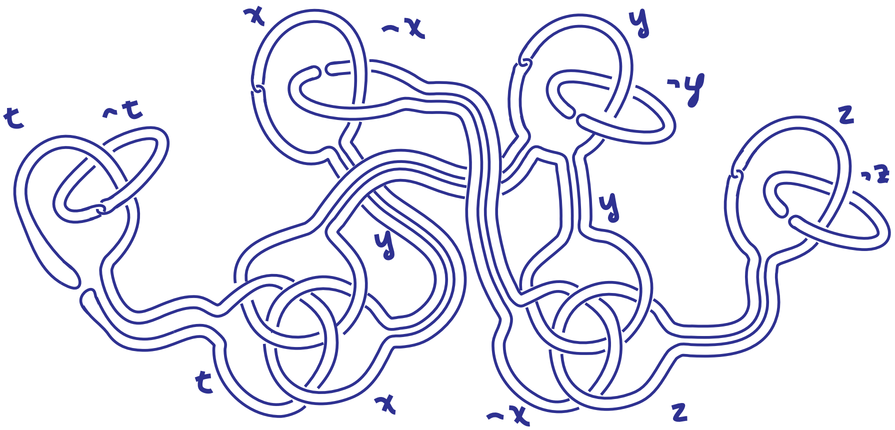


Will show: Φ is satisfiable $\iff$ unlinking number n

Φ satisfiable $\implies$ unlinking number n

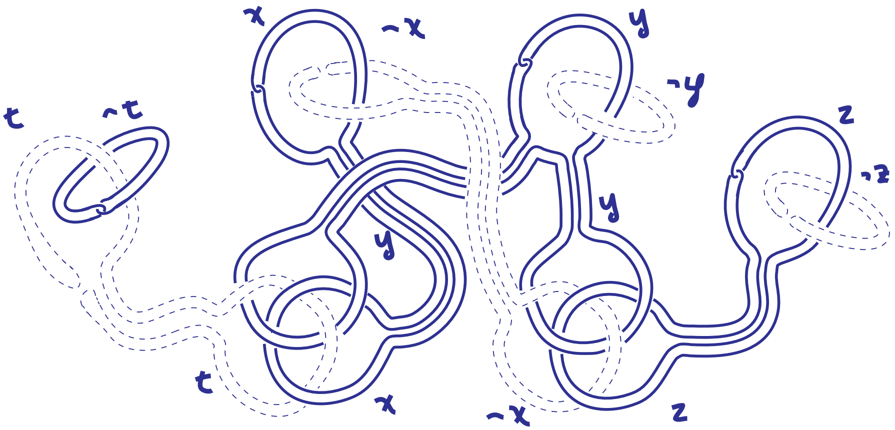
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Φ is satisfiable, unclasp TRUE components (n crossing changes).

Φ satisfiable $\implies$ unlinking number n

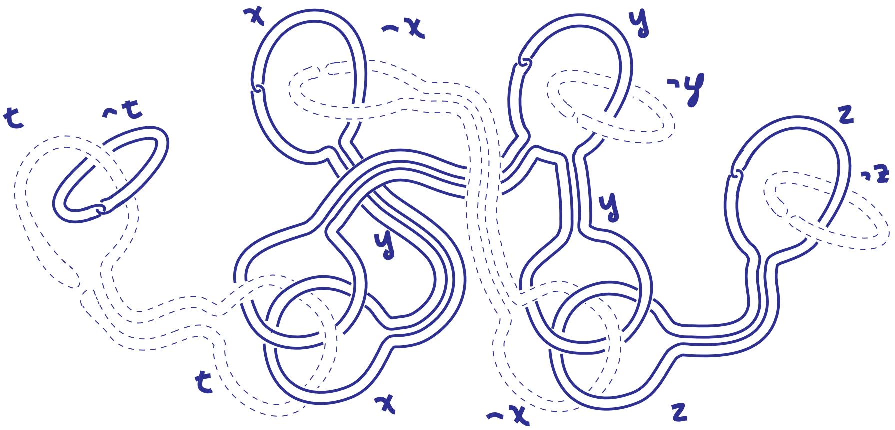
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



The TRUE components are an unlink, push to side.

Φ satisfiable $\implies$ unlinking number n

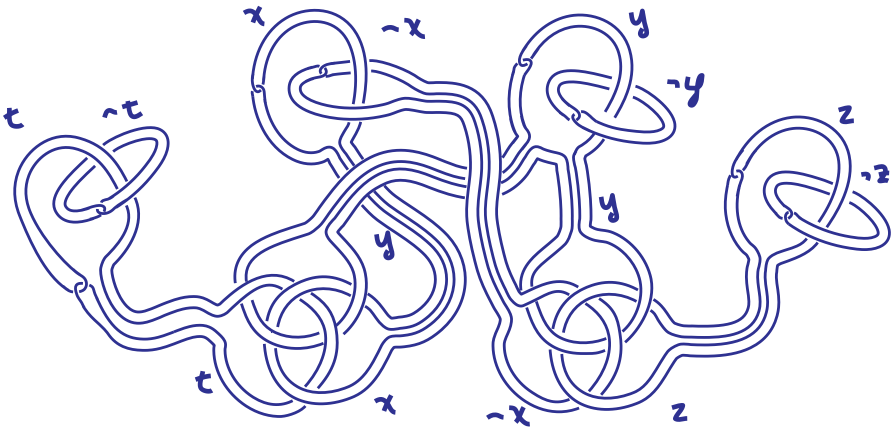
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



What remains is also an unlink! $\implies$ unlinking number n.

unlinking number $n \implies \Phi$ satisfiable

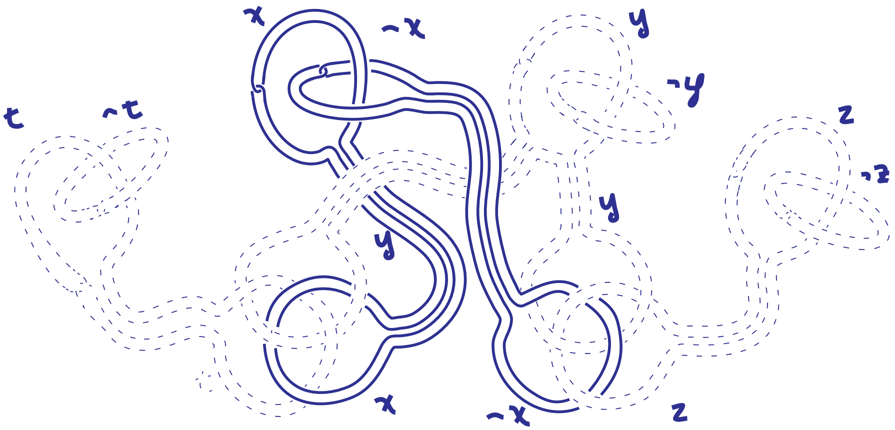
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Unlinking number $n \implies$

unlinking number $n \implies \Phi$ satisfiable

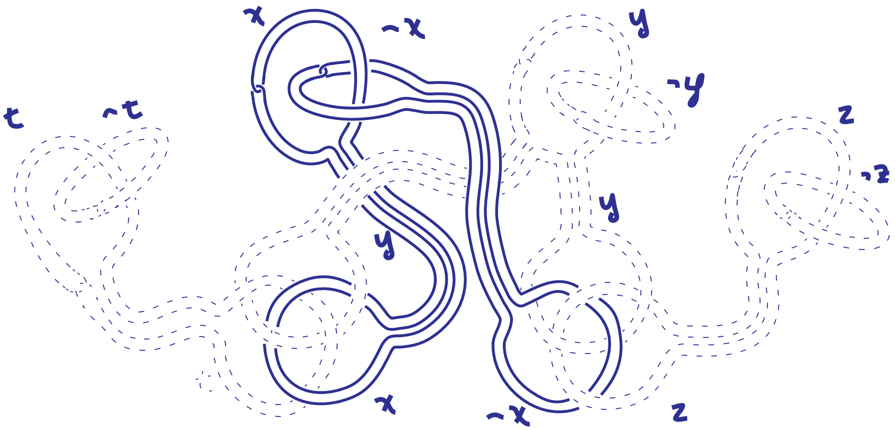
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Unlinking number $n \implies$ each variable gets a crossing change.

unlinking number $n \implies \Phi$ satisfiable

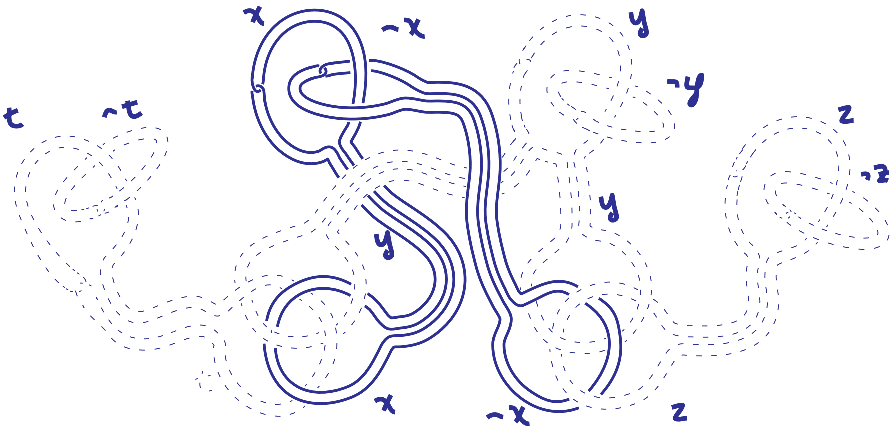
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Crossing change affects **either** x or $\neg x$ (not both).

unlinking number $n \implies \Phi$ satisfiable

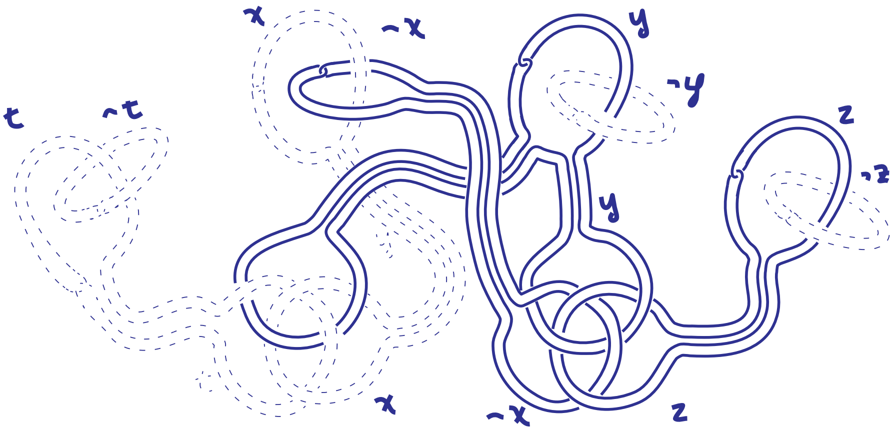
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Call the changed components TRUE

unlinking number $n \implies \Phi$ satisfiable

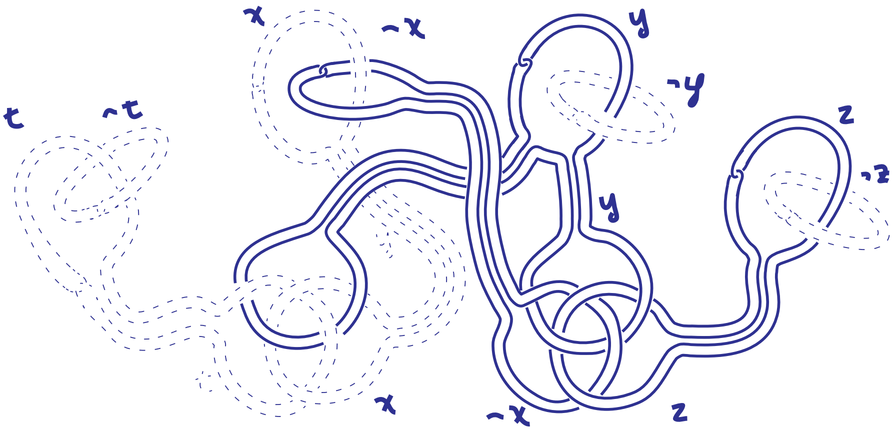
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Every Borromean clause has a changed crossing .

unlinking number $n \implies \Phi$ satisfiable

$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$

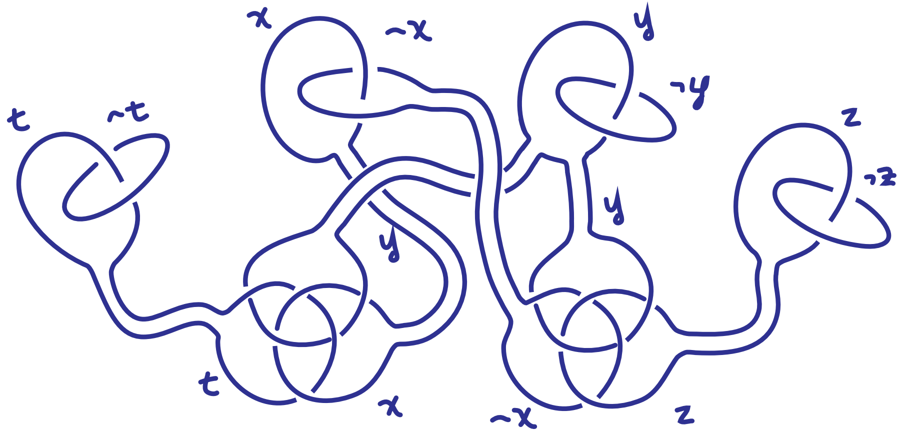


Every Borromean clause has a changed crossing $\implies \Phi$ satisfiable.

Reidemeister Defect is NP-hard

Reidemeister Defect is NP-hard

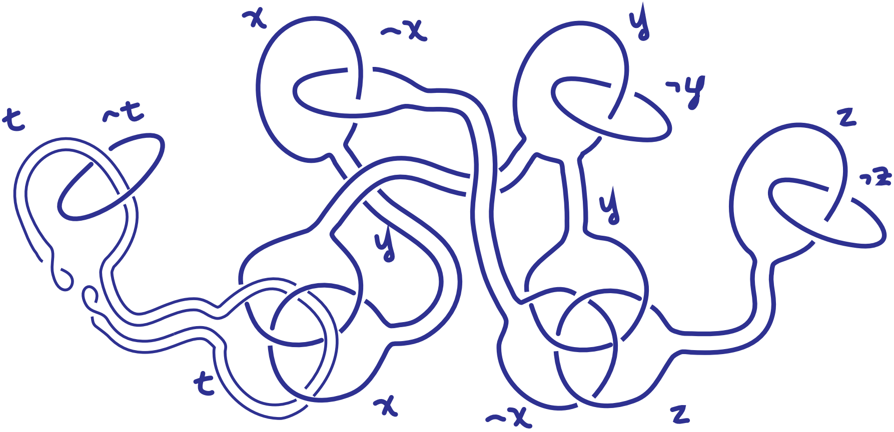
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Again, a very similar construction.

Reidemeister Defect is NP-hard

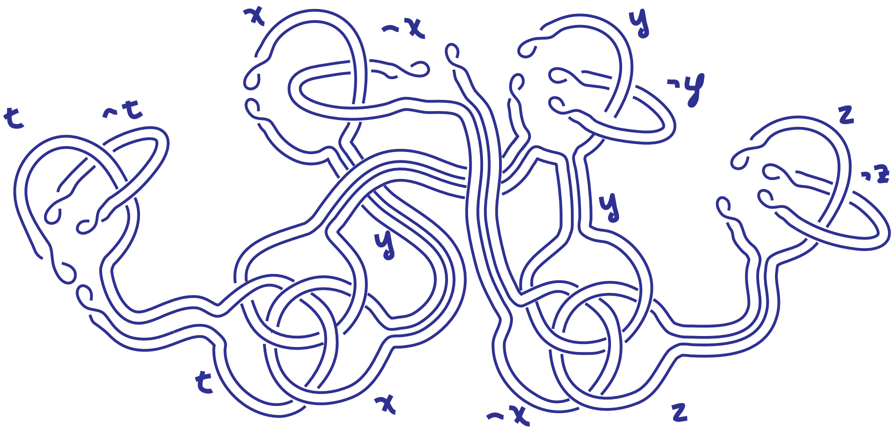
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



But replace each component with a *twisted unknot.*

Reidemeister Defect is NP-hard

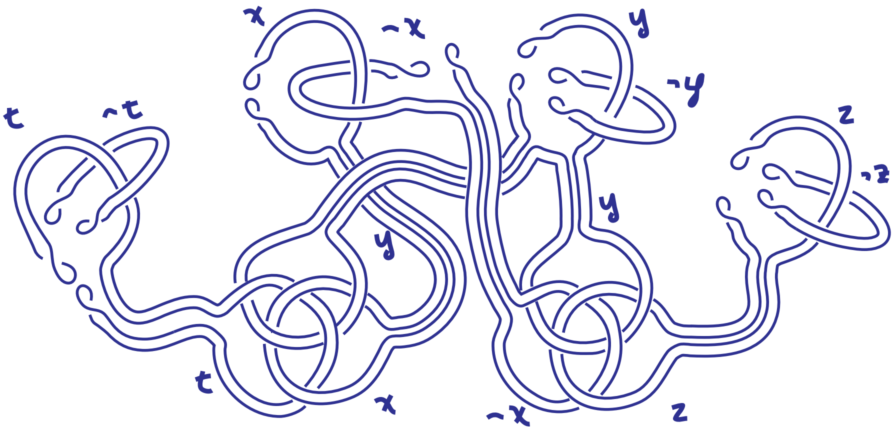
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



But replace each component with a *twisted unknot.*

Reidemeister Defect is NP-hard

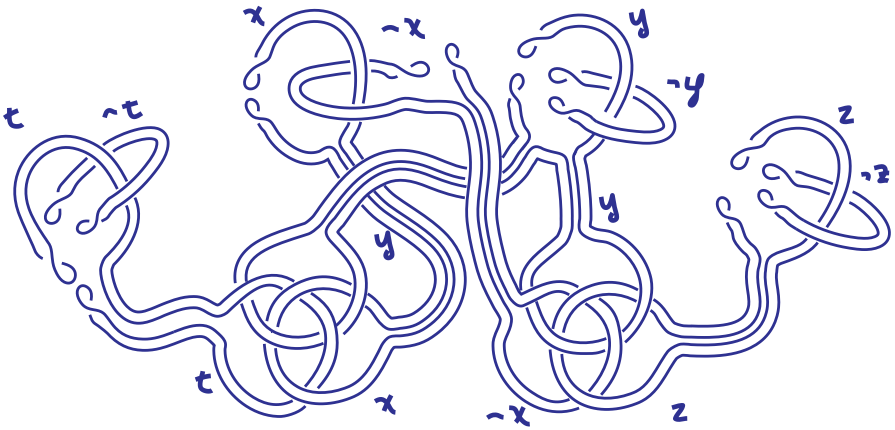
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



This is a diagram of an unlink.

Reidemeister Defect is NP-hard

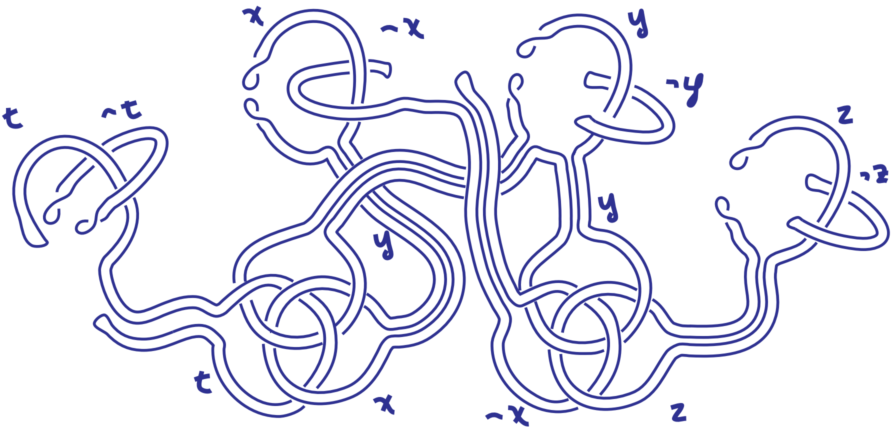
$$\Phi = (t \lor x \lor y) \land (\neg x \lor y \lor z)$$



Show: Φ is satisfiable $\iff$ Can trivialize diagram with deficit $= n$.

Φ satisfiable $\implies$ trivialize with deficit $= n$.

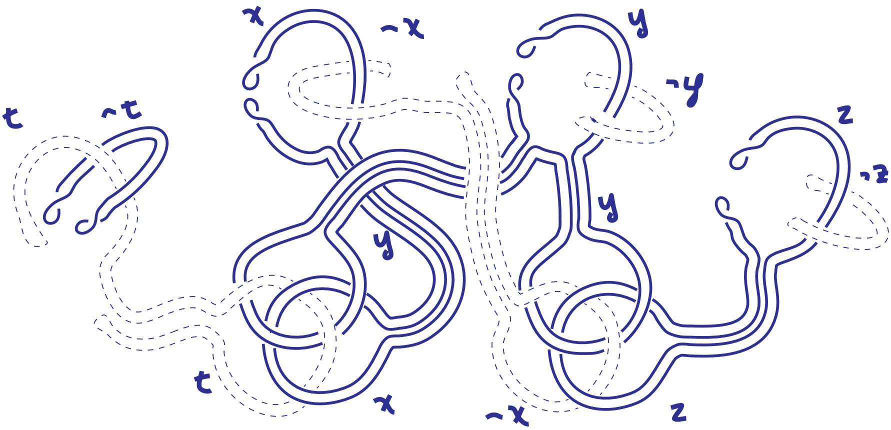
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



Φ is satisfiable, untwist ends of TRUE components, cost deficit n.

Φ satisfiable $\implies$ trivialize with deficit $= n$.

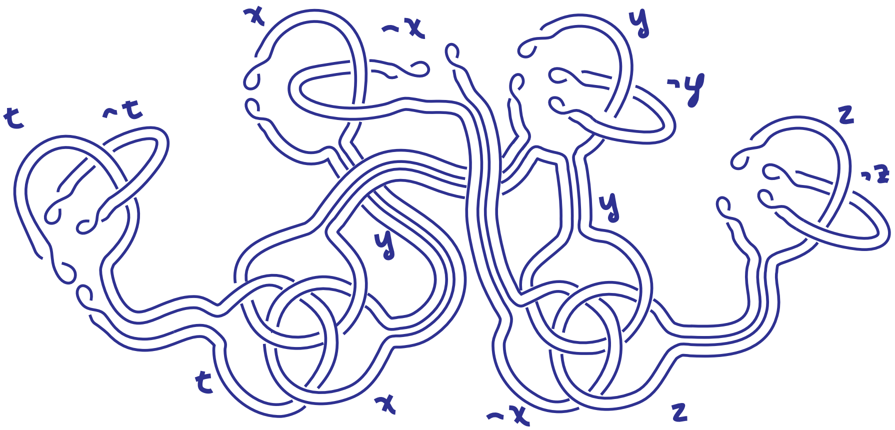
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg \, x \ \vee \ y \ \vee \ z)$$



What remains can be trivialized with *no* additional deficit.

Trivialize with deficit $= n \implies \Phi$ satisfiable.

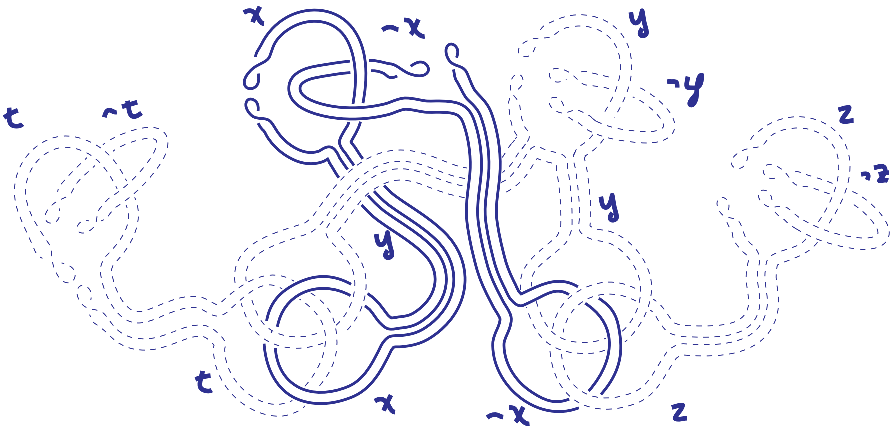
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Assume deficit $= n$.

Trivialize with deficit $= n \implies \Phi$ satisfiable.

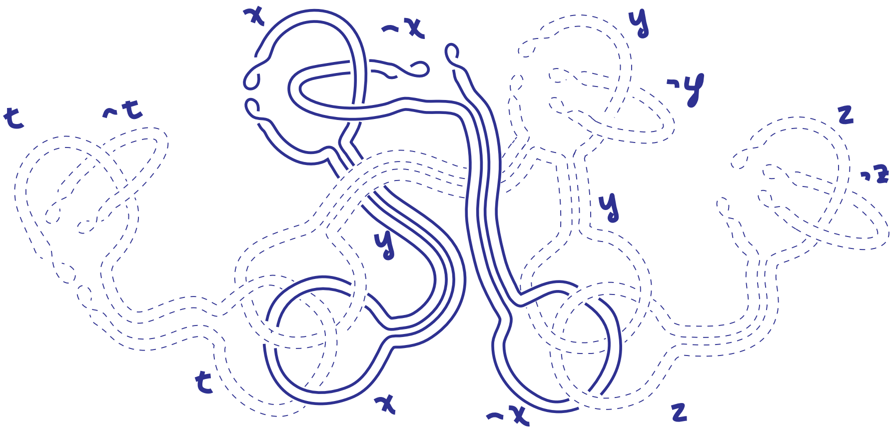
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Deficit $= n \implies$ each variable gets deficit 1.

Trivialize with deficit $= n \implies \Phi$ satisfiable.

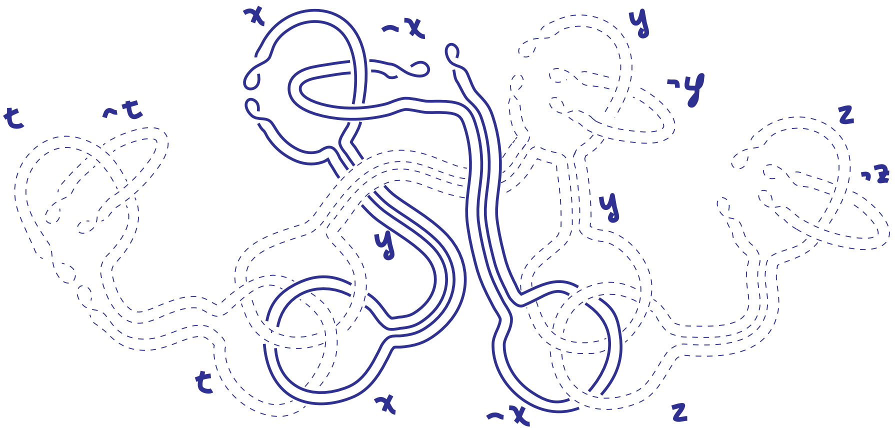
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Deficit move involves **either** x or $\neg x$ (not both).

Trivialize with deficit $= n \implies \Phi$ satisfiable.

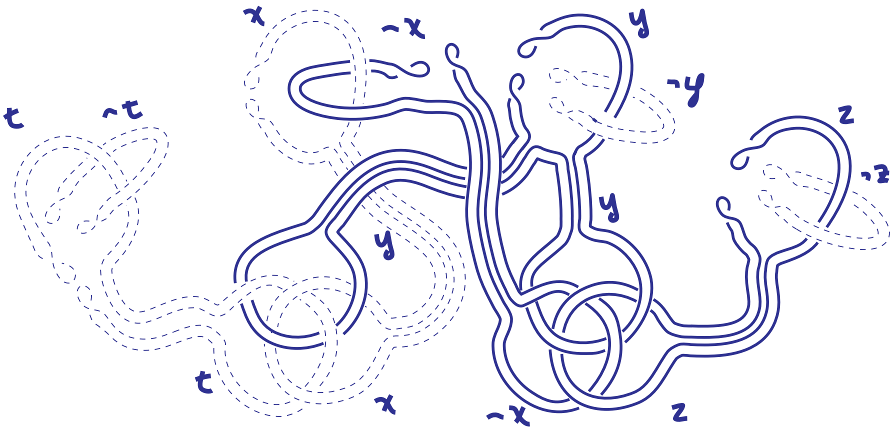
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



Call the component involved TRUE

Trivialize with deficit $= n \implies \Phi$ satisfiable.

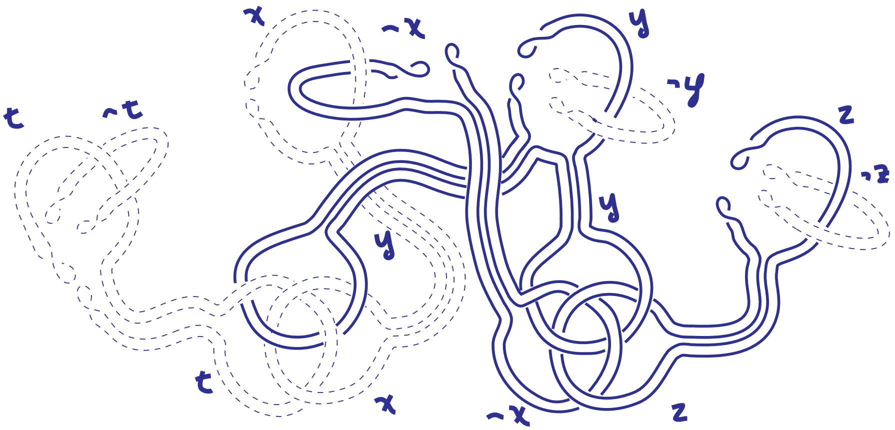
$$\Phi = (t \ \vee \ x \ \vee \ y) \ \wedge \ (\neg\, x \ \vee \ y \ \vee \ z)$$



Every Borromean clause has defict > 0.

Trivialize with deficit $= n \implies \Phi$ satisfiable.

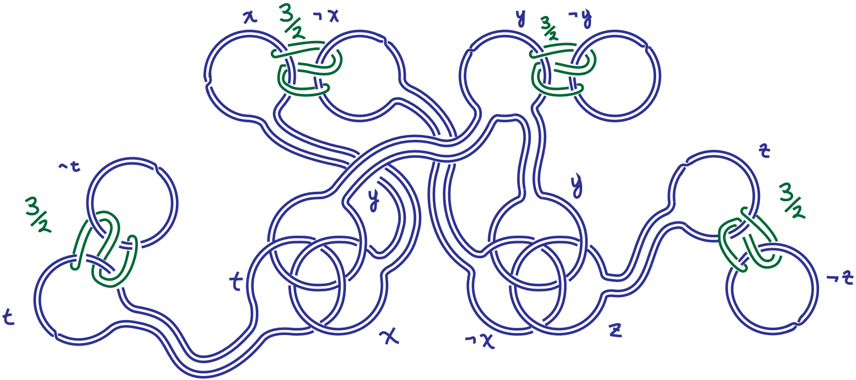
$$\Phi = (t \vee x \vee y) \wedge (\neg x \vee y \vee z)$$



$\implies \Phi$ satisfiable.

$\textsc{Embed}_{2\to 3}$ is NP-hard

$\text{EMBED}_{2\to 3}$ is NP-hard :



Uses a cabled link and **Dehn surgery**.

Open Questions:

	Knots	Links
TRIVIALITY	NP, co-NP[a]	NP
TRIVIAL SUB-LINK	n/a	NP-complete
UNLINKING NUMBER	?	NP-hard
REIDEMEISTER DEFECT	NP	NP-complete
REIDEMEISTER DISTANCE	?	NP-hard
3-MANIFOLD EMBEDS IN S^3	NP[b]	NP-hard

[a] Kuperberg; Lackenby; [b] Schleimer

Questions:

1. Is UNLINKING/UNKNOTTING NUMBER decidable?
2. Are UNLINKING NUMBER, REIDEMEISTER DISTANCE and EMBED$_{2\to 3}$ in NP?
3. Are UNLINKING (UNKNOTTING) NUMBER and REIDEMEISTER DISTANCE/DEFECT NP-hard for a single component?

Thanks!