Knots in $\mathbb{S}^3$ and minimal surfaces in $\mathbb{B}^4$

joint work with Marc Soret

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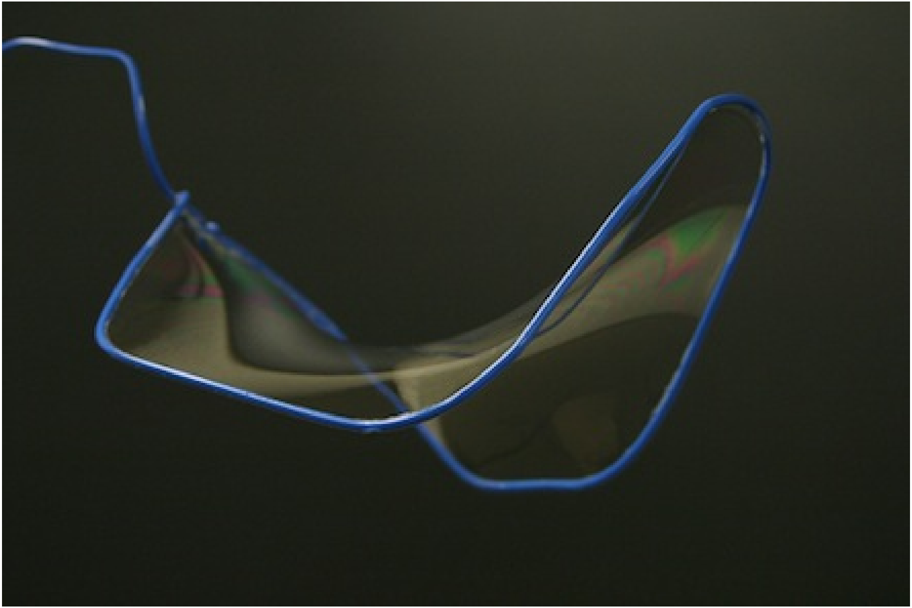
Université de Tours, France

Institut Henri Poincaré, June 22th, 2018



Paul Laurain, Image des maths

Minimal surfaces in $\mathbb{R}^4$

critical point for the area in any deformation with compact support

$$\frac{d(area(S_t))}{dt}\Big|_{t=0} = 0$$

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EXEMPLE. Complex curves in $\mathbb{C}^2 = \mathbb{R}^4$.

Ribbon knots

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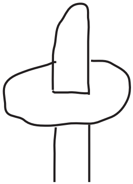

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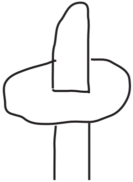
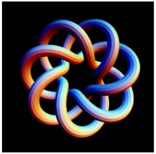


ribbon singularities

TH (Hass, 1983): a knot in $\mathbb{S}^3$ is ribbon iff it bounds an embedded minimal disk Δ in $\mathbb{B}^4$

REMARK. Harmonic parametrization ==> the restriction of $d(0,.)$ to Δ has no local maxima.

Torus knots



$K(3,7)$ torus knot
In $\mathbb{R}^3$, the parameter goes N times around a circle C in a vertical plane while C rotates p times around Oz.

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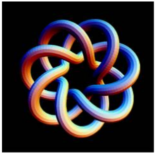

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In $\mathbb{R}^3$, the parameter goes N times around a circle C in a vertical plane while C rotates p times around Oz. In $\mathbb{S}^3$,

$$K(N,p) : \mathbb{S}^1 \longrightarrow \mathbb{S}^3$$

$$e^{i\theta} \mapsto (\frac{1}{\sqrt{2}}e^{Ni\theta}, \frac{1}{\sqrt{2}}e^{pi\theta})$$

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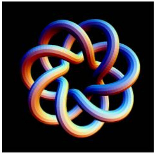
Algebraic curves



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$$C_{N,p} = \{(z_1, z_2)\ z_1^p = z_2^N\}$$

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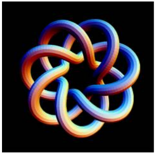

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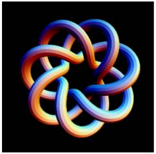

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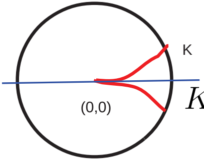
ex: cusp $z_1^3 = z_2^2$ (drawn in $\mathbb{R}^2$!)



$$K(N,p) = C_{N,p} \cap \mathbb{S}^3_\epsilon$$

NB. $(0,0)$ is a branch point; $C_{N,p}$ is not a smooth near $(0,0)$ but it has a tangent plane

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For ϵ small enough, the type of the knot does not depend on ϵ. There is a homeomorphism

$$Cone(\mathbb{S}^3_\epsilon, K_\epsilon) \cong (\mathbb{B}^4, F(\mathbb{D}))$$

WHO ARE THE KNOTS OF BRANCH POINTS OF MINIMAL DISKS??? CAN THEY BE ALL THE KNOTS??????

Constructing the knot

RECALL $-->$ Coordinate functions of a minimal surfaces are harmonic. So

Each of the 4 components of the minimal disk is a series in $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$. We truncate each component by larger and larger powers of r: as soon as we get something injective, we can stop and we have the knot type.

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SIMPLEST CASE. We can stop at the lowest order term of each of the 4 components.

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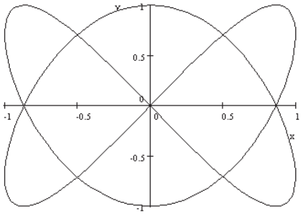
- $p \neq q$ Lissajous toric knot

Lissajous toric knots

Lissajous curve $C_{q,p,\phi}$ in a vertical plane

$$t \mapsto (\sin qt, \cos(p\theta + \phi))$$



Type I: $(\sin(2t), \cos(3t))$, $0 \le t \le 2\pi$

http://mathserver.neu.edu
/ bridger/U170/Lissajous/Lissajous.pdf

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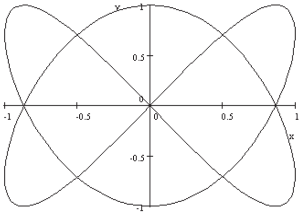

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- along $C_{q,p,\phi}$ while
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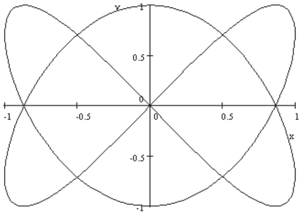

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Proposition

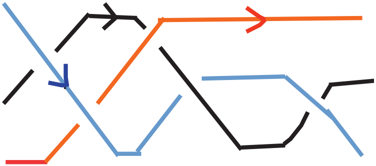
Up to mirror symmetry the knot type does not depend on the phase ϕ.

Billiard knots in a square solid torus

Christoph Lamm (PhD in the late 1990's, this chapter on arxiv in 2012): billiard knots in a square solid torus

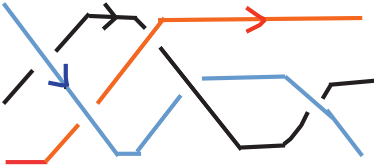
$$V = [0,1]^3/(0,y,z) \cong (1,y,z)$$

Braids



N points connected by N strands.

Braids



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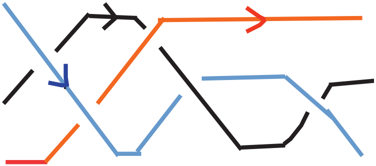

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sign of the crossing points

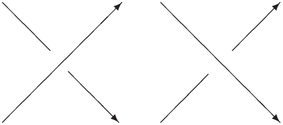

Fig. 2 — Sign of a crossing point

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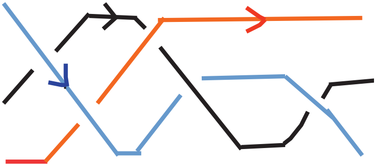

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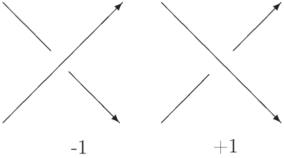

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Form a group B_N generated by $\sigma_1, \ldots, \sigma_{N-1}$
σ_i switches the i-th and $i+1$-th strand with relations

$$|i-j| \geq 2 ==> \sigma_i \sigma_j = \sigma_j \sigma_i$$

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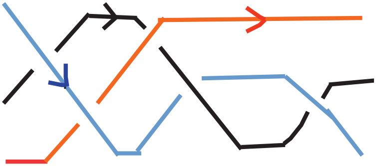

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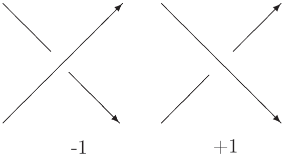

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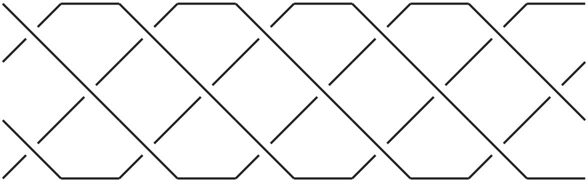
$$\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^2$$

Braid of the (N, q) torus knot



$N = 4, q = 5$

$$\Big(\prod_{1 \leq 2i+1 \leq N-1} \sigma_{2i+1} \prod_{2 \leq 2i \leq N-1} \sigma_{2i} \Big)^q$$

The braid $B_{N,q,p}$

We work with the knot in the $3D$-cylinder $\mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{R}^2$

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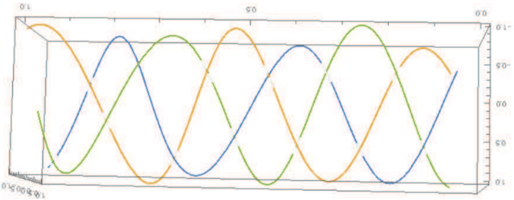
We derive the braid $B_{N,q,p,\phi}$ which represents the knot

The first 2 coordinates are the same as for torus knots



$N = 3$, $q = 7$, $p = 5$

Periodic case

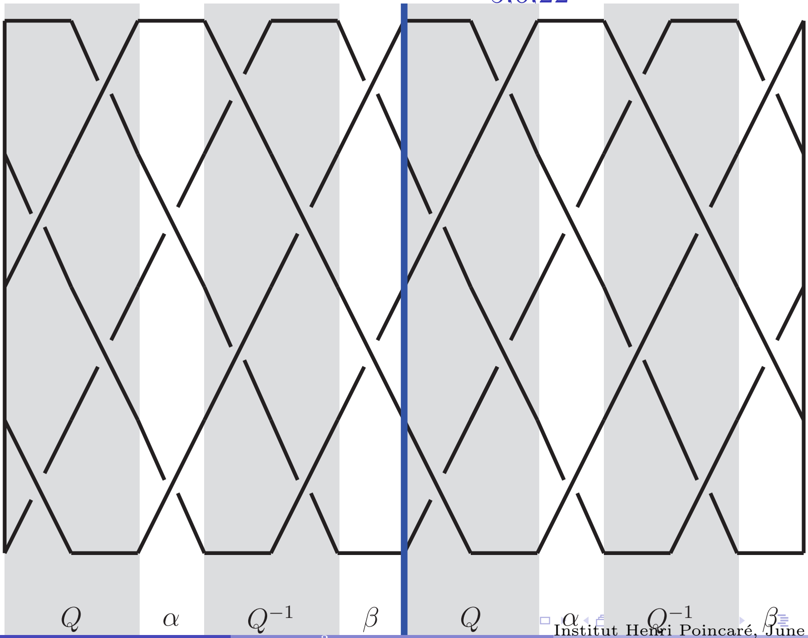
$$B_{N,dq,dp} = B^d_{N,q,p}$$

==> we assume:

- the numbers p and q are mutually prime
- q is odd

. Note: if $d > 1$, the knot $K(N, q, p)$ is periodic.

The braid $B_{5,6,22}$



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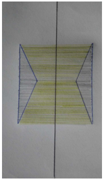

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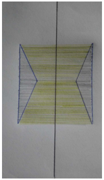

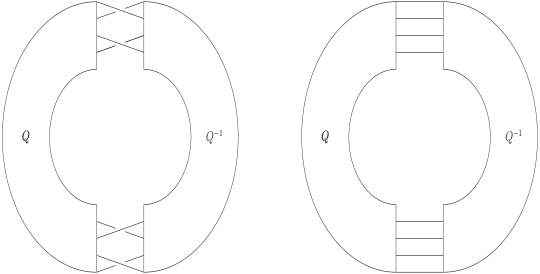

$N-1$ half-twist tangles connecting Q and Q^{-1}; replace them by $N-1$ tangles and get a N-component link L which is symmetric w.r.t. a plane and bounds N ribbon disks which intersect in ribbon singularities.

Compare and contrast with torus knots

Proposition (Soret-V., 2016)

For N, q, mutually prime, $K(N, q, q+N)$ is trivial.

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$g_4(K)$ = smallest genus of a surface bounded by K in $\mathbb{B}^4$.

Theorem (Kronheimer-Mrowka)

The 4-genus of the (N, d)-torus knot $K(N, d)$ is

$$g_4(K(N,d)) = \frac{(N-1)(d-1)}{2}$$

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Proposition (Soret-V., 2016)

Let $d = gcd(p, q)$. Then

$$g_4(K(N,q,p)) \leq \frac{(N-1)(d-1)}{2}$$

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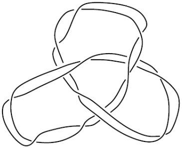
$$z \mapsto (z^6, z^{15} + z^{17})$$

all 6 strands are distinct.

Cable knot: a $(3,17)$ torus knot around the $(2,5)$ torus knot inside its tubular neighbourhood.



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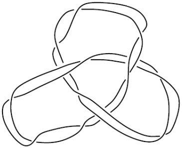
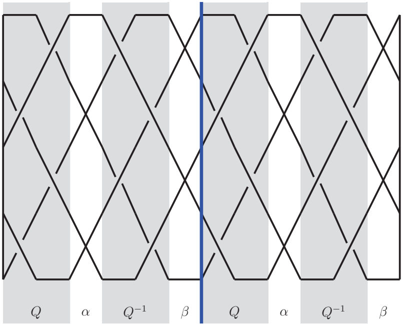


Similarly, we can cable Lissajous toric knots.
PROBLEM: when does the cable come from a minimal disk?

Second situation: critical phases

Note: unlike the cabling, this has no counterpart for algebraic knots. Let $(N, q) = (N, p) = 1$.

$$(e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \phi)) \text{ with singular crossing points}$$



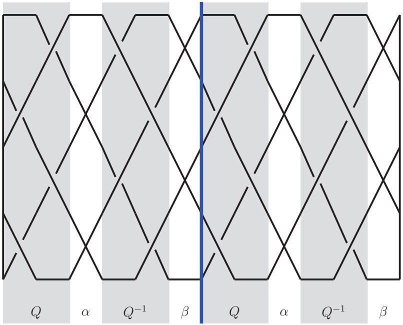
$$(r^N e^{Ni\theta}, r^q \sin(q\theta), r^p \cos(p\theta + \phi) + r^a \cos(a\theta + \beta)) \qquad a > p$$

Regular points are unchanged; singular parts are replaced by $\alpha_{N,q,a}$ and $\beta_{N,q,a}$.

Second situation: critical phases

Note: unlike the cabling, this has no counterpart for algebraic knots. Let $(N, q) = (N, p) = 1$.

$(e^{Ni\theta}, \sin(q\theta), \cos(p\theta + \phi))$ with singular crossing points



$$(r^N e^{Ni\theta}, r^q \sin(q\theta), r^p \cos(p\theta + \phi) + r^a \cos(a\theta + \beta)) \qquad a > p$$

Regular points are unchanged; singular parts are replaced by $\alpha_{N,q,a}$ and $\beta_{N,q,a}$. We get a minimal knot. Iterate?

A problem: when we want *minimal* disks

$$(z^N, Im(z^q), Re(z^p e^{i\phi}))$$

A problem: when we want *minimal* disks

$$(z^N, Im(z^q), Re(z^p e^{i\phi}))$$

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$$(z^N + \bar{h}(z), Im(z^q), Re(z^p e^{i\phi}))$$

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$$(z^N, Im(z^q), Re(z^p e^{i\phi}))$$

minimal?

Let $w^N = z^N + \bar{h}(z)$

$$w = z + o(|z|)$$

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$$= (w^N, Im(w^q) + o(|w|^q), Re(w^p e^{i\phi}) + o(|w|^p))$$

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CONCLUSION: if we stop at the first order terms, the term $\bar{h}(z)$ does not matter; it may matter if we go to a higher order.

Conclusion

CONJECTURE: not every knot is isotopic to a minimal knot.
Reasons: the cosines which make up the knots have different order of magnitude, according to the rank of the term where they appear.

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Theorem (Soret-V., 2015)

Let K be a knot. There exist n_1, n_2, n_3, n_4 integers, ϕ, ψ, ϵ rational numbers such that K is isotopic to the knot given in $\mathbb{R}^3$ by

- $x = \cos(2\pi n_1 t)$
- $y = \cos\left(2\pi n_2 t + \phi\right) + \epsilon \cos(2\pi n_3 t + \psi)$
- $z = \cos(2\pi n_4 t + \tau)$

Appendix: software

Feed the data of the braid into KnotPlot which computes the Alexander and Jones polynomial of the knot. If the crossing number is not too large, identify it in the Rohlfsen or Hoste-Thistlethwaite tables.

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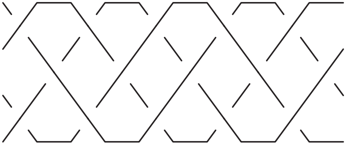
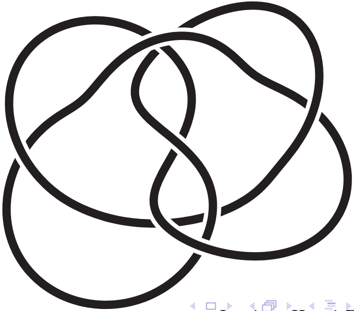
Feed the data of the braid into KnotPlot which computes the Alexander and Jones polynomial of the knot. If the crossing number is not too large, identify it in the Rohlfsen or Hoste-Thistlethwaite tables.$- - -- >$ exemple of a non fibered prime minimal knot (Soret-V. 2011), 9_{46} representing $K(4, 13, 5)$





THANK

YOU!!